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An Effective Stochastic Semiclassical Theory for the Gravitational Field

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Assuming that the mechanism proposed by Gell-Mann and Hartle works as a mechanism for decoherence and classicalization of the metric field, we formally derive the form of an effective theory for the gravitational field in a semiclassical regime. This effective theory takes the form of the usual semiclassical theory of gravity, based on the semiclassical Einstein equation, plus a stochastic correction which accounts for the backreaction of the lowest order matter stress-energy fluctuations.

1. INTRODUCTION

In the semiclassical theory of gravity, the gravitational field is treated classically, but the matter fields are quantum. The key equation of the theory is the semiclassical Einstein equation, a generalization of the Einstein equation where the expectation value of the stress-energy tensor of quantum matter fields is the source of curvature.

One expects that semiclassical gravity could be derived from a fundamental quantum theory of gravity as a certain approximation, but in the absence of such a fundamental theory, the scope and limits of the semiclassical theory are not very well understood. It seems clear, nevertheless, that it should not be valid unless gravitational fluctuations are negligibly small. This condition may break down when the matter stress-energy has appreciable quantum fluctuations, since one would expect that fluctuations in the stress-energy of matter would induce gravitational fluctuations [1]. A number of examples have been recently studied, both in cosmological and flat space-times, where, for some states of the matter fields, the stress-energy tensor

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has significant fluctuations [2]. To account for such fluctuations, it is necessary to extend the semiclassical theory of gravity.

To address this problem or analogous problems in quantum mechanics or quantum field theory, different approaches have been adopted in the literature. The present paper attempts to unify, at least conceptually, two of these approaches in a formal derivation of an effective theory for the gravitational field in the semiclassical regime. The common feature of these two approaches is the idea of viewing the metric field as the system of interest and the matter fields as being part of its environment. This idea was first proposed by Hu [3] in the context of semiclassical cosmology. Both approaches make use of the influence functional formalism introduced by Feynman and Vernon [4] to deal with a system–environment interaction in a full quantum theory. In this formalism, the integration of the environmental variables in a path integral yields the influence functional, from which one can define an effective action for the dynamics of the system [5–12].

The first of these two approaches has been extensively used in the literature, not only in the framework of semiclassical cosmology [6–8, 13–16], but also in the context of analogous semiclassical regimes for systems of quantum mechanics [9, 11, 17] and of quantum field theory [12, 18–21]. It makes use of the closed time path (CTP) functional technique due to Schwinger and Keldysh [22]. This is a path-integral technique designed to obtain expectation values of field operators in a direct way [23]. In the semiclassical regime, a tree-level approximation is performed in the path integrals involving the system variables. In this approximation, the equation of motion for the expectation value of the system field operator is the semiclassical equation, which can be directly derived from the effective action of Feynman and Vernon [6, 12, 13, 15, 16, 20]. When computing this effective action perturbatively up to quadratic order in its variables, one usually finds some imaginary terms which do not contribute to the semiclassical equation. The key point of this approach is the formal identification of the contribution of such terms to the influence functional with the characteristic functional of a Gaussian stochastic source. Assuming that in the semiclassical regime this stochastic source interacts with the system variables, and thus these become stochastic variables, equations of the Langevin type are derived for these variables. However, since this approach relies on a purely formal identification, doubts can be raised on the physical meaning of the derived equations.

The second approach is based on the description of the transition from quantum to classical behavior in the framework of the consistent histories formulation of a quantum theory. The consistent histories formulation, proposed by Griffiths [24] and developed by Omnès [25] and Gell-Mann and Hartle [26, 27], was designed to deal with quantum closed (i.e., isolated) systems. It is thus believed to be an appropriate approach to quantum cosmology, where the quantum system is the whole universe. The main goal of this formulation is the study of the conditions under which a set of quantum mechanical variables become decoherent, which means that these variables can be described in a probabilistic way [26–30]. When the closed system consists of a distinguished subsystem (the "system," which is also often called an "open system") interacting with its environment, Gell-Mann and Hartle proposed a mechanism for decoherence and classicalization of suitably coarse-grained system variables [26, 27]. This approach allows one to evaluate the probability distribution functional associated with such decoherent variables and, under some approximations, to derive effective quasiclassical equations of motion of the Langevin type for such variables [26–28, 31, 32].

In Section 2, we show that that these two approaches can in fact be related. In this way, we see that, on the one hand, the second approach sheds light on the physical meaning of the first one. On the other hand, the first approach provides a tool for computing effective Langevin-type equations to the second one. A large portion of this section consists in reformulating the mechanism for decoherence and classicalization of Gell-Mann and Hartle in the language of the CTP functional formalism.

In Section 3, we use the results of this analysis to formally derive effective equations of motion for the gravitational field in a semiclassical regime. This derivation relies heavily on the results of the previous section. We find that, in the semiclassical regime, gravity might be described by a background metric, solution of the semiclassical Einstein equation, plus some stochastic metric perturbations. The equation for these perturbations, the semiclassical Einstein–Langevin equation, is seen to incorporate the effect of the lowest order matter stress-energy fluctuations on the gravitational field.

In this paper we use the $(+\ +\ +)$ sign conventions and the abstract index notation of ref. 33, and we work in units in which $c = \hbar = 1$.

2. EFFECTIVE EQUATIONS OF MOTION FROM ENVIRONMENT-INDUCED CLASSICALIZATION

2.1. The CTP Functional Formalism for a System–Environment Interaction

We start this section by sketching the CTP functional formalism [22] applied to a system–environment interaction and its relation with the influence functional formalism of Feynman and Vernon [4]. For more detailed reviews of the CTP functional formalism, see refs. 23 and 16, and for the influence functional formalism of Feynman and Vernon, see refs. 5–12. For simplicity, we shall work in this section with a model of quantum mechanics, but all the formalism can also be formally applied to field theory. It is instructive to maintain in this section the explicit dependence on $\hbar$. Let us consider a model of quantum mechanics which describes the interaction of two subsystems: one, called the "system," with coordinates q, and the other, called the "environment," with coordinates Q.[3] We write the action for this model as $S[q, Q] = S_s[q] + S_{se}[q, Q]$.[4] Let $\hat{q}(t)$ and $\hat{Q}(t)$ be the Heisenberg-picture coordinate operators, which are assumed to be self-adjoint, i.e., $\hat{q}^\dagger = \hat{q}$ and $\hat{Q}^\dagger = \hat{Q}$, and let $\hat{q}^{\rm S}$ and $\hat{Q}^{\rm S}$ be the corresponding Schrödinger-picture operators. Suppose that we are only interested in describing the physical properties of system observables from some initial time t_i until some final time $t_f > t_i$. Working in the Schrödinger picture, the state of the full system (i.e., system plus environment) at the initial time $t = t_i$ will be described by a density operator $\hat{\rho}^{\rm S}(t_i)$. Let $\{|q, Q\rangle^{\rm S}\}$ be the basis of eigenstates of the operators $\hat{q}^{\rm S}$ and $\hat{Q}^{\rm S}$. The matrix elements of the initial density operator in this basis will be written as $\rho(q, Q; q', Q'; t_i) \equiv {}^{\rm S}\langle q,Q|\hat{\rho}^{\rm S}(t_i)|q',Q'\rangle^{\rm S}$. For simplicity, we shall assume that the initial density operator can be factorized as $\hat{\rho}^{\rm S}(t_i) = \hat{\rho}^{\rm S}_s(t_i) \otimes \hat{\rho}^{\rm S}_e(t_i)$ in such a way that its matrix elements in coordinate representation can be written as $\rho(q, Q; q', Q'; t_i) = \rho_s(q, q'; t_i)\rho_e(Q, Q'; t_i)$. However, the formalism can be generalized to the most general case of a nonfactorizable initial density operator [36, 37, 26]. We are interested in computing expectation values of operators related to the system variables only, for times t between t_i and t_f. The dynamics of the system in this sense can be completely characterized by the knowledge of the whole family of Green functions of the system. Working in the Heisenberg picture, these Green functions can be defined as expectation values of products of $\hat{q}(t)$ operators. These Green functions can be derived from a CTP generating functional in which only the system variables are coupled to external sources $j_+(t)$ and $j_-(t)$ [6, 12, 13, 16, 19, 20]. This CTP generating functional can be written as the following path integral[5]:

[3] Even if, in order to simplify the notation, we do not write indices in these coordinates, q and Q have to be understood as representing an arbitrary number of degrees of freedom (which, in particular, can be an infinite number of degrees of freedom).

[4] We shall assume that the action $S[q, Q]$ is the one that appears in the path integral formulas for the model, which, in general, does not need to coincide with the classical action for the model [34, 35].

[5] A way of generalizing the formalism to a nonfactorizable initial density operator consists in the following refs. 36 and 26. One writes the initial density matrix in coordinate representation as $\rho(q, Q; q', Q'; t_i) = \rho_s(q, q'; t_i)\, \rho_{se}(q, Q; q', Q'; t_i)$, where ρ_s is chosen in such a way that $\int dq\, \rho_s(q, q; t_i) = 1$. Then, the CTP generating functional can be written as (2.1), with

$$
\begin{aligned}
\exp\left\{\frac{i}{\hbar} S_{\rm eff}[q_+, q_-]\right\}
&\equiv \int \mathcal{D}[Q_+]\, \mathcal{D}[Q_-]\, \rho_{se}(q_{+_i}, Q_{+_i}; q_{-_i}, Q_{-_i}; t_i)\, \delta(Q_{+_f} - Q_{-_f}) \\
&\quad \times \exp\left\{\frac{i}{\hbar}\left(S[q_+, Q_+] - S[q_-, Q_-]\right)\right\}
\end{aligned}
$$

$$Z[j_+, j_-] = \int \mathcal{D}[q_+]\,\mathcal{D}[q_-]\,\rho_s(q_{+_i}, q_{-_i}; t_i)\,\delta\,(q_{+_f} - q_{-_f})$$

$$\times \exp\left\{\frac{i}{\hbar}\left(S_{\text{eff}}[q_+, q_-] + \hbar \int dt\, j_+ q_+ - \hbar \int dt\, j_- q_-\right)\right\} \tag{2.1}$$

with

$$S_{\text{eff}}[q_+, q_-] \equiv S_s[q_+] - S_s[q_-] + S_{\text{IF}}[q_+, q_-] \tag{2.2}$$

where S_{IF} is the influence action of Feynman and Vernon, which is defined in terms of the influence functional $\mathcal{F}_{\text{IF}}$ as

$$\mathcal{F}_{\text{IF}}[q_+, q_-] \equiv \exp\left\{\frac{i}{\hbar} S_{\text{IF}}[q_+, q_-]\right\}$$

$$\equiv \int \mathcal{D}[Q_+]\,\mathcal{D}[Q_-]\,\rho_e(Q_{+_i}, Q_{-_i}; t_i)\,\delta(Q_{+_f} - Q_{-_f})$$

$$\times \exp\left\{\frac{i}{\hbar}\left(S_{se}[q_+, Q_+] - S_{se}[q_-, Q_-]\right)\right\} \tag{2.3}$$

We shall call $S_{\text{eff}}[q_+, q_-]$ the effective action of Feynman and Vernon. In these expressions we use the notation $q_{+_i} \equiv q_+(t_i)$, $q_{+_f} \equiv q_+(t_f)$, $Q_{+_i} \equiv Q_+(t_i)$, $Q_{+_f} \equiv Q_+(t_f)$, and similarly for q_- and Q_-. All the integrals in t, including those that would define the actions $S_s[q]$ and $S_{se}[q, Q]$ in terms of the corresponding Lagrangians, have to be understood as integrals between t_i, and t_f. The CTP generating functional has the properties

$$Z[j, j] = 1, \qquad Z[j_-, j_+] = Z^*[j_+, j_-], \qquad \left|Z[j_+, j_-]\right| \leq 1 \tag{2.4}$$

From this generating functional, we can derive the following Green function for the system:

$$\langle \tilde{T}[\hat{q}(t'_1) \cdots \hat{q}(t'_s)]T[\hat{q}(t_1) \cdots \hat{q}(t_r)]\rangle$$

$$= \left.\frac{\delta Z[j_+, j_-]}{i\delta j_+(t_1) \cdots i\delta j_+(t_r)(-i)\delta j_-(t'_1) \cdots (-i)\delta j_-(t'_s)}\right|_{j_\pm = 0} \tag{2.5}$$

where $t_1, \ldots, t_r, t'_1, \ldots, t'_s$ are all between t_i and t_f, and T and T mean, respectively, time and anti-time ordering. The expectation value is taken in the Heisenberg-picture state corresponding to the Schrödinger-picture state described by $\hat{\rho}^{S}(t_i)$ at the initial time $t = t_i$. The influence functional (2.3) can actually be interpreted as a CTP generating functional for quantum variables Q coupled to classical time-dependent sources $q(t)$ through the action $S_{se}[q, Q]$ [38]. Let us consider the quantum theory for the variables Q in presence of classical sources $q(t)$ corresponding to this action, and assume that the initial Schrödinger-picture state for the quantum variables Q is described by the density operator $\hat{\rho}_e^{S}(t_i)$. For this theory, let $\hat{\mathcal{U}}[q](t, t')$ be the unitary time-evolution operator, which can be formally written as $\hat{\mathcal{U}}[q](t, t') = \mathrm{T}\exp[-(i/\hbar)\int_{t'}^{t} dt'' \hat{H}^S[q](t'')]$, for $t > t'$, where $\hat{H}^S[q](t)$ is the Hamiltonian operator in the Schrödinger picture. This Hamiltonian operator depends on t as a function of $q(t)$ and its derivative $\dot{q}(t)$, and this gives a functional dependence on q in the operator $\hat{\mathcal{U}}$. It is easy to see that [26, 27, 7, 11, 12, 36]

$$\begin{aligned}\mathcal{F}_{\mathrm{IF}}[q_+, q_-] &= \mathrm{Tr}[\hat{\rho}_e^{S}(t_i)\hat{\mathcal{U}}^{\dagger}[q_-](t_f, t_i)\hat{\mathcal{U}}[q_+](t_f, t_i)] \\ &= \langle \hat{\mathcal{U}}^{\dagger}[q_-](t_f, t_i)\hat{\mathcal{U}}[q_+](t_f, t_i)\rangle_{\hat{\rho}_e^{S}(t_i)} \end{aligned} \tag{2.6}$$

where we use $\langle\,\cdot\,\rangle_{\hat{\rho}_e^{S}(t_i)}$ to denote an expectation value in the state described by $\hat{\rho}_e^{S}(t_i)$. From this expression, it follows that the influence functional satisfies

$$\mathcal{F}_{\mathrm{IF}}[q, q] = 1, \qquad \mathcal{F}_{\mathrm{IF}}[q_-, q_+] = \mathcal{F}^{*}_{\mathrm{IF}}[q_+, q_-], \qquad \left|\mathcal{F}_{\mathrm{IF}}[q_+, q_-]\right| \leq 1 \tag{2.7}$$

or, equivalently, in terms of the influence action,

$$S_{\mathrm{IF}}[q, q] = 0, \qquad S_{\mathrm{IF}}[q_-, q_+] = -S^{*}_{\mathrm{IF}}[q_+, q_-], \qquad \mathrm{Im}\, S_{\mathrm{IF}}[q_+, q_-] \geq 0 \tag{2.8}$$

and similar properties follow for $S_{\mathrm{eff}}[q_+, q_-]$. A decoherence functional for the system, where the environment variables have been completely integrated out, can now be introduced as the functional Fourier transform of the CTP generating functional in the external sources:

$$Z[j_+, j_-] \equiv \int \mathcal{D}[q_+]\mathcal{D}[q_-]\mathcal{D}[q_+, q_-] \exp\left[i \int dt\, (j_+ q_+ - j_- q_-) \right] \tag{2.9}$$

that is, from (2.1) we have that

$$\mathcal{D}[q_+, q_-] = \rho_s(q_{+_i}, q_{-_i}; t_i)\, \delta(q_{+_f} - q_{-_f}) \exp\left\{ \frac{i}{\hbar} S_{\mathrm{eff}}[q_+, q_-] \right\} \tag{2.10}$$

In the consistent histories approach to quantum mechanics, $\mathcal{D}[q_+, q_-]$ is known as the decoherence functional for fine-grained histories of the system [26–29, 31, 32].

The environment of a system has to be understood as characterized by all the quantum degrees of freedom which can affect the dynamics of the system, but which are "not accessible" in the observations of that system. This environment includes in general an "external" environment (variables representing other particles, or, in the context of field theory, other fields) and an "internal" environment (some degrees of freedom which, from the fundamental quantum theory point of view, would be associated to the same physical object as the "system" variables, but which are not directly probed in our observations of the system) [39, 25]. For instance, a problem which has been studied using the influence functional method is that of quantum Brownian motion [4, 5, 9–11, 17, 26–28, 31, 32, 36, 37, 40]. In this problem, one is interested in the dynamics of a macroscopic particle interacting with a medium composed of a large number of other particles. In this example, one considers that the only "observable" system degree of freedom is the center-of-mass position of the macroscopic particle, whereas the remaining microscopic degrees of freedom of the macroscopic particle are considered as environmental variables. Such "internal" environment degrees of freedom, and also those of the particles of the medium (the "external" environment), are usually modeled as an infinite set of harmonic oscillators. In the context of field theory, one would typically consider as "inaccessible" to the observations the modes of the field of interest with characteristic momenta higher than some cutoff momentum [41, 12, 18]. In the case of the gravitational field, this has been considered by Whelan [42] in a toy model designed to investigate the decoherence mechanism for gravity.

It is convenient at this stage to distinguish between these two kinds of environmental variables, so let Q represent the coordinates of the "external" environment (the coordinates of "other particles") and $q_{\rm U}$ the "unobservable system" coordinates (the coordinates of the "internal" environment). As before, q will represent the "true" system coordinates. One could now simply replace Q by $(Q, q_{\rm U})$ in the previous expressions. However, for convenience, we shall do the integrations in the environmental variables in two steps. The action of the full system will be now written as $S[q, q_{\rm U}, Q]$, and, as before, we shall assume a totally factorizable initial density operator $\hat{\rho}^{\rm S}(t_i) = \hat{\rho}^{\rm S}_s(t_i) \otimes \hat{\rho}^{\rm S}_{\rm U}(t_i) \otimes \hat{\rho}^{\rm S}_e(t_i)$, which leads to an initial density matrix in coordinate representation of the form $\rho(q, q_{\rm U}, Q; q', q'_{\rm U}, Q'; t_i) = \rho_s(q, q'; t_i)\rho_{\rm U}(q_{\rm U}, q'_{\rm U}; t_i)\rho_e(Q, Q'; t_i)$ (notice that we are now using the index e for the "external" environment). Such a factorization is based on the assumption that the interactions among the three subsystems can be neglected for times $t \leq t_{\rm i}$. Unfortunately, in most situations, this assumption does not seem to be very physically reasonable, especially for the "true" system–"internal" environment interactions. One would need to consider the generalization of the formalism to a nonfactorizable initial density operator mentioned above and the analysis would be more complicated. We start by defining

$$\exp\left\{\frac{i}{\hbar}\left(S_s^{\text{eff}}[q_+] - S_s^{\text{eff}}[q_-] + S_{se}^{\text{eff}}[q_+, Q_+; q_-, Q_-]\right)\right\}$$
$$\equiv \int \mathcal{D}[q_{U+}]\mathcal{D}[q_{U-}]\rho_U(q_{U+_i}, q_{U-_i}; t_i)\delta(q_{U+_f} - q_{U-_f})$$
$$\times \exp\left\{\frac{i}{\hbar}\left(S[q_+, q_{U+}, Q_+] - S[q_-, q_{U-}, Q_-]\right)\right\} \quad (2.11)$$

where the effective action for the system $S_s^{\text{eff}}[q]$ is chosen to be real and local. Notice that the effective action $S_{se}^{\text{eff}}[q_+, Q_+; q_-, Q_-]$ has analogous properties to those of S_{IF} in (2.8). We introduce now an effective influence functional and an effective influence action as

$$\mathcal{F}_{\text{IF}}^{\text{eff}}[q_+, q_-] \equiv \exp\left\{\frac{i}{\hbar} S_{\text{IF}}^{\text{eff}}[q_+, q_-]\right\}$$
$$\equiv \int \mathcal{D}[Q_+]\mathcal{D}[Q_-]\rho_e(Q_{+_i}, Q_{-_i}; t_i)\delta(Q_{+_f} - Q_{-_f})$$
$$\times \exp\left\{\frac{i}{\hbar} S_{\text{se}}^{\text{eff}}[q_+, Q_+; q_-, Q_-]\right\} \quad (2.12)$$

With these definitions, the effective action of Feynman and Vernon, $S_{\text{eff}}[q_+, q_-]$, which appears in expression (2.1) can be written as

$$S_{\text{eff}}[q_+, q_-] \equiv S_s^{\text{eff}}[q_+] - S_s^{\text{eff}}[q_-] + S_{\text{IF}}^{\text{eff}}[q_+, q_-] \quad (2.13)$$

Note that, since $S_{\text{eff}}[q_+, q_-]$ satisfies the same properties as S_{IF} in (2.8), it follows from the last expression that $S_{\text{IF}}^{\text{eff}}$ has also these properties.

2.2. The "Naive" Semiclassical Approximation

The usual "naive" semiclassical approximation for the system variables consists in performing a "tree-level" approximation in the path integrals involving the q variables in expression (2.1) [6, 12, 13, 15, 16, 20]. Therefore, the CTP generating functional is approximated by

$$Z[j_+, j_-] \simeq \exp\left\{\frac{i}{\hbar}\left(S_{\text{eff}}[\bar{q}_+^{(0)}[j], q_-^{(0)}[j]] + \hbar \int dt\, j_+ \bar{q}_+^{(0)}[j] - \hbar \int dt\, j_- \bar{q}_-^{(0)}[j]\right)\right\} \tag{2.14}$$

where $\bar{q}_\pm^{(0)}[j] \equiv \bar{q}_\pm^{(0)}[j_+, j_-]$ are solutions of the classical equations of motion for the action $S_{\text{eff}}[q_+, q_-] + \hbar \int dt\, j_+ q_+ - \hbar \int dt\, j_- q_-$, that is,

$$\frac{\delta S_{\text{eff}}[\bar{q}_+^{(0)}, \bar{q}_-^{(0)}]}{\delta q_\pm(t)} = \mp \hbar\, j_\pm(t) \tag{2.15}$$

which satisfy the boundary condition $\bar{q}_+^{(0)}(t_f) = \bar{q}_-^{(0)}(t_f)$. Whenever this approximation is valid, we can see from (2.14), (2.15), and (2.5) that $\langle \hat{q}(t)\rangle \simeq q^{(0)}(t)$, with $q^{(0)} \equiv \bar{q}_+^{(0)}[j_+ = j_- = 0] = \bar{q}_-^{(0)}[j_+ = j_- = 0]$, that is, $q^{(0)}(t)$ is a solution of the two equivalent equations

$$\left.\frac{\delta S_{\text{eff}}[q_+, q_-]}{\delta q_+(t)}\right|_{q_+ = q_- = q^{(0)}} = 0, \qquad \left.\frac{\delta S_{\text{eff}}[q_+, q_-]}{\delta q_-(t)}\right|_{q_+ = q_- = q^{(0)}} = 0 \tag{2.16}$$

One can see that these two equations are actually the same equation, and that this equation is real. This is the semiclassical equation for the system variables. In a naive way, one would think that, when the above semiclassical approximation is valid, the system would behave as a classical system described by the coordinate functions $q^{(0)}(t)$, i.e., that one could substitute the description of the system in terms of the operators $\hat{q}(t)$ by a classical description in terms of the functions $q^{(0)}(t)$. However, one can see from (2.14), (2.15), and (2.5) that, in general,

$$\langle \mathrm{T}[\hat{q}(t_1') \cdots \hat{q}(t_s')]\mathrm{T}[\hat{q}(t_1) \cdots \hat{q}(t_r)]\rangle \neq q^{(0)}(t_1) \cdots q^{(0)}(t_r) q^{(0)}(t_1') \cdots q^{(0)}(t_s') \tag{2.17}$$

Thus, in general, whenever the above approximations are valid, we can only interpret the solutions of the semiclassical equation as representing the expectation value of the operators $\hat{q}(t)$.

2.3. Further Coarse-Graining and Decoherence

Decoherence takes place in a set of quantum mechanical variables when the quantum interference effects are (in general, approximately) suppressed in the description of the properties of a physical system which are associated to that variables. When this happens, such decoherent variables can be described in an effective probabilistic way. In the Heisenberg picture, we will say that a set of variables decohere when the description in terms of the operators corresponding to these variables can be replaced by an effective description in terms of a set of classical random variables, in the sense that the quantum Green functions for such operators become approximately equal to the moments of the classical random variables. For the Green functions (2.5), it is easy to see that this would hold in an exact way if the CTP generating functional (2.1) depended on the sources $j_\pm$ only as a functional $\Phi_q[j_+ - j_-]$ of the difference $j_+ - j_-$, or, equivalently, if the decoherence functional (2.9) could be written as $\mathcal{D}[q_+, q_-] = \mathcal{P}_q[q_+]\,\delta[q_+ - q_-]$. However, in practice, one finds that this condition is usually too strong to be satisfied, even in an approximate way [26–29, 31, 32, 42]. One needs to introduce further coarse-graining in the system degrees of freedom in order to achieve decoherence. Let us then introduce coarse-grained system operators, which correspond to imprecisely specified values of the system coordinates. In the Heisenberg picture, such operators can be defined as

$$\hat{q}_c(t) \equiv \sum_{\bar{q}} \bar{q}\hat{P}_{\bar{q}}(t) \tag{2.18}$$

where $\hat{P}_{\bar{q}}(t)$ is a set of projection operators, labeled by some variables $\bar{q}$ (these are often discrete variables), of the form

$$\hat{P}_{\bar{q}}(t) = \int dq\, dq_{\mathrm{U}}\, dQ\; \gamma(q - \bar{q}) |q, q_{\mathrm{U}}, Q, t\rangle\langle q, q_{\mathrm{v}}, Q, t| \tag{2.19}$$

Here $\{|q, q_{\mathrm{U}}, Q, t\rangle\}$ is the basis of eigenstates of the operators $\hat{q}(t)$, $\hat{q}_{\mathrm{U}}(t)$, and $\hat{Q}(t)$, and γ is a real function. We shall assume coarse grainings of characteristic sizes σ, that is, such that the function $\gamma(q - \bar{q})$ vanishes or has negligible values for q outside a cell $I_{\bar{q}}$ of sizes σ centered around $\bar{q}$. This means that

$$\int dq\; \gamma(q - \bar{q}) f(q) \simeq \int_{I_{\bar{q}}} dq\; \gamma(q - \bar{q}) f(q) \tag{2.20}$$

for any function $f(q)$. In addition, the function γ must be chosen in such a way that the set of projection operators is (at least, approximately) exhaustive and mutually exclusive, which means that

$$\sum_{\bar{q}} \hat{P}_{\bar{q}}(t) = \hat{I}, \qquad \hat{P}_{\bar{q}}(t)\hat{P}_{\bar{q}'}(t) = \delta_{\bar{q}\bar{q}'}\hat{P}_{\bar{q}}(t) \tag{2.21}$$

where $\hat{I}$ is the identity operator. For specific examples of operators satisfying the above properties in an exact or in an approximate way, see refs. 31 and 32.

Next, we can introduce a family of decoherence functions for coarse-grained histories of the system [26–32]. In order to do so, let us consider a set $\{t_1, \ldots, t_N\}$ of N instants of time, such that $t_k < t_{k+1}$, $k = 0, \ldots, N$, with $t_0 \equiv t_i$ and $t_{N+1} \equiv t_f$. Introducing two sets of values of $\bar{q}$ associated to such set of instants, $\{\bar{q}_+\} \equiv \{\bar{q}_{+1}, \ldots, \bar{q}_{+N}\}$ and $\{\bar{q}_-\} \equiv \{\bar{q}_{-1}, \ldots, \bar{q}_{-N}\}$, the decoherence function for this pair of "coarse-grained histories" of the system is defined as

$$\mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1, \ldots, t_N)} \equiv \mathrm{Tr}[\hat{P}_{\bar{q}_{+N}}(t_N) \cdots \hat{P}_{\bar{q}_{+1}}(t_1)\hat{\rho}\hat{P}_{\bar{q}_{-1}}(t_1) \cdots \hat{P}_{\bar{q}_{-N}}(t_N)] \tag{2.22}$$

where $\hat{\rho}$ is the density operator describing the state of the entire system (system plus environment) in the Heisenberg picture ($\mathcal{D}_c$ is often called decoherence "functional" in the literature, but for each set $\{t_1, \ldots, t_N\}$, this is actually a function of $2N$ variables). These decoherence functions can be written in a path integral form as

$$\mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1, \ldots, t_N)} = \int \mathcal{D}[q_+]\mathcal{D}[q_-] \prod_{k=1}^{N} \gamma(q_+(t_k) - \bar{q}_{+k})\gamma(q_-(t_k) - \bar{q}_{-k})\mathcal{D}[q_+, q_-] \tag{2.23}$$

where $\mathcal{D}[q_+, q_-]$ is the decoherence functional for fine-grained histories of the system (2.9).

From the definition (2.22) and the properties (2.21), one can show that these decoherence functions have the properties

$$\sum_{\{\bar{q}_+\}} \sum_{\{\bar{q}_-\}} \mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\}) = 1, \qquad \mathcal{D}_c(\{\bar{q}_-\}, \{\bar{q}_+\}) = \mathcal{D}_c^*(\{\bar{q}_+\}, \{\bar{q}_-\}) \tag{2.24}$$

and that the diagonal elements of the decoherence functions (the values of those functions in the limit $\bar{q}_{-k} \to \bar{q}_{+k}$) are positive. For $N > 1$, we can also see that, if we divide the set $\{t_1, \ldots, t_N\}$ into a subset of $M < N$ instants, $\{t'_1, \ldots, t'_M\} \subset \{t_1, \ldots, t_N\}$, with $t'_1 < \cdots < t'_M$, and the subset of the remaining $L \equiv M - N$ instants, denoted as $\{t''_1, \ldots, t''_L\}$ [i.e., $\{t_1, \ldots, t_N\} = \{t'_1, \ldots, t'_M\} \cup \{t''_1, \ldots, t''_L\}$], then

$$\mathcal{D}_c(\{\bar{q}_+\}_M, \{\bar{q}_-\}_M)_{(t'_1, \ldots, t'_M)} = \sum_{\{\bar{q}_+\}_L} \sum_{\{\bar{q}_-\}_L} \mathcal{D}_c(\{\bar{q}_+\}_N, \{\bar{q}_-\}_N)_{(t_1, \ldots, t_N)} \tag{2.25}$$

with $\{\bar{q}_\pm\}_M \equiv \{\bar{q}_\pm(t'_1), \ldots, \bar{q}_\pm(t'_M)\}$, $\{\bar{q}_\pm\}_L \equiv \{\bar{q}_\pm(t''_1), \ldots, \bar{q}_\pm(t''_L)\}$, where we use the notation $\bar{q}_\pm(t_k) \equiv \bar{q}_{\pm k}$, for $k = 1, \ldots, N$, and $\{\bar{q}_\pm\}_N \equiv \{\bar{q}_{\pm 1}, \ldots, \bar{q}_{\pm N}\}$.

To make contact with the CTP formalism, let us introduce now, in analogy with (2.9), a family of generating functions for the coarse-grained system degrees of freedom as the following Fourier series:

$$Z_c(\{j_+\}, \{j_-\})_{(t_1,\ldots,t_N)} \equiv \sum_{\{\bar{q}+\}} \sum_{\{\bar{q}-\}} \mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1,\ldots,t_N)} \times \exp\left\{ i \sum_{k=1}^{N} (j_{+k}\bar{q}_{+k} - j_{-k}\bar{q}_{-k}) \right\} \tag{2.26}$$

where $\{j_\pm\} \equiv \{j_{\pm 1}, \ldots, j_{\pm N}\}$. Note that the properties (2.24) for the decoherence functions are equivalent to

$$Z_c(\{0\}, \{0\}) = 1, \qquad Z_c(\{j_-\}, \{j_+\}) = Z_c^*(\{j_+\}, \{j_-\}) \tag{2.27}$$

From the generating function (2.26), we can compute the Green functions

$$G_{c\, m_1\cdots m_s}^{n_1\cdots n_r}(t'_1, \ldots, t'_r; t''_1, \ldots, t''_s) \equiv \langle \tilde{\mathrm{T}}[\hat{q}_c^{m_1}(t''_1) \cdots \hat{q}_c^{m_s}(t''_s)] \mathrm{T}[\hat{q}_c^{n_1}(t'_1) \cdots \hat{q}_c^{n_r}(t'_r)] \rangle$$

with $n_1, \ldots, n_r, m_1, \ldots, m_s \in \mathbb{N}$, $\{t'_1, \ldots, t'_r\} \subseteq \{t_1, \ldots, t_N\}$, and $\{t''_1, \ldots, t''_s\} \subseteq \{t_1, \ldots, t_N\}$ (thus, $r, s \le N$):

$$G_{c\, m_1\cdots m_s}^{n_1\cdots n_r}(t'_1, \ldots, t'_r; t''_1, \ldots, t''_s) = \left. \frac{(-i\partial)^{n_1+\cdots+n_r+m_1+\cdots+m_s} Z_c(\{j_+\}, \{j_-\})_{(t_1,\ldots,t_N)}}{[\partial j_+(t'_1)]^{n_1} \cdots [\partial j_+(t'_r)]^{n_r} [-\partial j_-(t''_1)]^{m_1} \cdots [-\partial j_-(t''_s)]^{m_s}} \right|_{\{j_\pm\}=\{0\}} \tag{2.28}$$

where $j_\pm(t_k) \equiv j_{\pm k}$, for $k = 1, \ldots, N$. The property (2.25) can also be written in terms of the corresponding generating functions as

$$Z_c(\{j_+\}_M, \{j_-\}_M)_{(t'_1,\ldots,t'_M)} = Z_c(\{j_+\}_N, \{j_-\}_N)_{(t_1,\ldots,t_N)} \Big|_{\{j_\pm\}_L=\{0\}} \tag{2.29}$$

with the notation $\{j_\pm\}_M \equiv \{j_\pm(t'_1), \ldots, j_\pm(t'_M)\}$, and similarly for $\{j_\pm\}_L$ and $\{j_\pm\}_N$. Notice that this last property is consistent with (2.28), in the sense that, for instance, $G_c^{n_1 n_2}(t'_1, t'_2)$ can be equally computed either from $Z_c(\{j_+\}_2, \{j_-\}_2)_{(t'_1,t'_2)}$ or from $Z_c(\{j_+\}_N, \{j_-\}_N)_{(t_1,\ldots,t_N)}$, with $N > 2$.

Having introduced the coarse-grained description of the system in terms of the operators $\hat{q}_c(t)$, we can now sketch the decoherence mechanism for them. For the Green functions (2.28), one can show that the decoherence condition described above holds in an exact way if the generating function

(2.26) depends on the sources $j_{\pm k}$ only as a function of the differences $j_{+k} - j_{-k}$, i.e., as $\Phi_{\bar{q}}(\{j_+ - j_-\})_{(t_1,\ldots,t_N)}$. Then, introducing the Fourier series corresponding to $\Phi_{\bar{q}}$, we can write

$$
\begin{aligned}
& Z_c(\{j_+\}, \{j_-\})_{(t_1,\ldots,t_N)} \\
&= \Phi_{\bar{q}}(\{j_+ - j_-\})_{(t_1,\ldots,t_N)} \\
&\equiv \sum_{\{\bar{q}\}} \mathcal{P}_{\bar{q}}(\{\bar{q}\})_{(t_1,\ldots,t_N)} \exp\left\{ i \sum_{k=1}^{N} \bar{q}_k (j_{+k} - j_{-k}) \right\}
\end{aligned}
\tag{2.30}
$$

Note from the last expression that, if we interpret the function $\mathcal{P}_{\bar{q}}$ as the probability distribution for a set of random variables $\bar{q}_k$, $k = 1, \ldots, N$, associated to the instants t_k, then $\Phi_{\bar{q}}$ is the corresponding characteristic function. Therefore, from (2.28), we get

$$
\begin{aligned}
& G_{c\, m_1 \cdots m_s}^{n_1 \cdots n_r}(t'_1, \ldots, t'_r; t''_1, \ldots, t''_s) \\
&= \left. \frac{(-i\partial)^{n_1+\cdots+n_r+m_1+\cdots+m_s} \Phi_{\bar{q}}(\{j\})_{(t_1,\ldots,t_N)}}{[\partial j(t'_1)]^{n_1} \cdots [\partial j(t'_r)]^{n_r} [\partial j(t''_1)]^{m_1} \cdots [\partial j(t''_s)]^{m_s}} \right|_{\{j\}=\{0\}} \\
&= \sum_{\{\bar{q}\}} \mathcal{P}_{\bar{q}}(\{\bar{q}\})_{(t_1,\ldots,t_N)} \bar{q}^{n_1}(t'_1) \cdots \bar{q}^{n_r}(t'_r) \bar{q}^{m_1}(t''_1) \cdots \bar{q}^{m_s}(t''_s) \\
&\equiv \langle \bar{q}^{n_1}(t'_1) \cdots \bar{q}^{n_r}(t'_r) \bar{q}^{m_1}(t''_1) \cdots \bar{q}^{m_s}(t''_s) \rangle_c
\end{aligned}
\tag{2.31}
$$

where $\langle \cdot \rangle_c$ means statistical average of the random variables, and we use the notation $\bar{q}(t_k) \equiv \bar{q}_k$, $j(t_k) \equiv j_k$, for $k = 1, \ldots, N$. Note that if (2.30) is satisfied, then the property (2.29) reduces to

$$
\Phi_{\bar{q}}(\{j\}_M)_{(t'_1,\ldots,t'_M)} = \Phi_{\bar{q}}(\{j\}_N)_{(t_1,\ldots,t_N)} \Big|_{\{j\}_L=\{0\}}
\tag{2.32}
$$

or, equivalently,

$$
\mathcal{P}_{\bar{q}}(\{\bar{q}\}_M)_{(t'_1,\ldots,t'_M)} = \sum_{\{\bar{q}\}_L} \mathcal{P}_{\bar{q}}(\{\bar{q}\}_N)_{(t_1,\ldots,t_N)}
\tag{2.33}
$$

This last property is a necessary condition for the probabilistic interpretation (2.31) to be consistent.

The conditions for decoherence (2.30) can be written in terms of the corresponding decoherence functions as

$$
\mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1,\ldots,t_N)} = \mathcal{P}_{\bar{q}}(\{\bar{q}_+\})_{(t_1,\ldots,t_N)} \prod_{k=1}^{N} \delta_{\bar{q}_{+k}\bar{q}_{-k}}
\tag{2.34}
$$

These are actually the conditions for decoherence of coarse-grained system variables as stated in the consistent histories formulation of quantum mechanics [26–32]. Notice, from (2.21), that (2.34) is always satisfied for a single instant of time (i.e., when $N = 1$) [31].

We can now check that the interpretation of $\mathcal{P}_{\bar{q}}$ as a probability function is actually correct. From the second of the properties (2.24), we have that $\mathcal{P}^*_{\bar{q}}(\{\bar{q}\}) = \mathcal{P}_{\bar{q}}(\{\bar{q}\})$, i.e., $\mathcal{P}_{\bar{q}}$ is real. Since the diagonal elements of the decoherence functions are positive, $\mathcal{P}_{\bar{q}}(\{\bar{q}\})$ is also positive. These two properties of $\mathcal{P}_{\bar{q}}(\{\bar{q}\})_{(t_1,\ldots,t_N)}$, together with (2.33), are enough to guarantee that it can be properly interpreted as the probability distribution for a set of random variables associated to the instants $t_1, \ldots, t_N$. From the first of the relations (2.24), which yields $\Sigma_{\{\bar{q}\}} \mathcal{P}_{\bar{q}}(\{\bar{q}\}) = 1$, it follows that this probability distribution is normalized.

In practice, the conditions for decoherence described above will be usually only satisfied in an approximate way. Approximate decoherence is typically achieved through a mechanism which was proposed by Gell-Mann and Hartle [26, 27]. To see how this works, note that, if we assume coarse-grainings of characteristic sizes σ [see (2.20)], and using (2.10), we can write the decoherence function (2.23) as

$$
\begin{aligned}
&\mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1,\ldots,t_N)} \\
&\quad \simeq \int_{\{I_{\bar{q}_+}\},\{I_{\bar{q}_-}\}} \mathcal{D}[q_+^{(0)}]\mathcal{D}[q_-^{(0)}] \prod_{k=1}^{N} \mathcal{D}[q_+^{(k)}]\mathcal{D}[q_-^{(k)}]\rho_s(q_{+i}^{(0)}, q_{-i}^{(0)}; t_i)\delta(q_{+f}^{(N)} - q_{-f}^{(N)}) \\
&\quad \times \delta(q_+^{(k-1)}(t_k) - q_+^{(k)}(t_k))\delta(q_-^{(k-1)}(t_k) - q_-^{(k)}(t_k)) \\
&\quad \times \gamma(q_+^{(k)}(t_k) - \bar{q}_{+_k})\gamma(q_-^{(k)}(t_k) - \bar{q}_{-_k}) \prod_{k=0}^{N} \exp\left\{\frac{i}{\hbar} S_{\rm eff}[q_+^{(k)}, q_-^{(k)}]\right\}
\end{aligned}
\tag{2.35}
$$

where each path integration $\int \mathcal{D}[q_\pm^{(k)}]$, for $k = 0, \ldots, N$, is over paths $q_\pm^{(k)}(t)$ with $t \in [t_k, t_{k+1}]$, with $t_0 \equiv t_i$ and $t_{N+1} \equiv t_f$, and we have used a notation to indicate that these paths are restricted to pass through the cells $I_{\bar{q}_{\pm_k}}$ at the instants t_k, for $k = 1, \ldots, N$. From (2.13), the modulus of each factor $\exp\{(i/\hbar)S_{\rm eff}[q_+^{(k)}, q_-^{(k)}]\}$ in the last expression is $\exp\{-(1/\hbar)\ \mathrm{Im}\ S_{\rm IF}^{\rm eff}[q_+^{(k)}, q_-^{(k)}]\}$. Then, if for every $k = 0, \ldots, N$, Im $S_{\rm IF}^{\rm eff}[q_+^{(k)}, q_-^{(k)}]$, which is always positive or zero, is much larger than $\hbar$ whenever the differences $\left|q_+^{(k)} - q_-^{(k)}\right|$ are larger than some "cutoff" sizes $d^{(k)}$, the integrand in (2.35) will be only nonnegligible for $\left|q_+^{(k)} - q_-^{(k)}\right| \le d^{(k)}$. If the characteristic sizes σ of the coarse-graining satisfy $\sigma \gg d^{(k)}$, then the "off-diagonal" elements of $\mathcal{D}_c(\{\bar{q}_+\}, \{\bar{q}_-\})_{(t_1,\ldots,t_N)}$ are negligible and one has approximate decoherence [26, 27]. We should stress that $S_{\rm IF}^{\rm eff}[q_+, q_-]$ is the result of integrating out both the "external" environment degrees of freedom and also the system degrees of freedom which are "not accessible" to the observations (the "internal" environment). In general, these two integrations play an important role in the achievement of this sufficient condition for approximate decoherence. A characterization of the degree of approximate decoherence has been given in ref. 31 (see also refs. 29, 28).

Typically, $d^{(k)}$ can be estimated in terms of $\Delta t_k \equiv t_{k+1} - t_k$. When this is the case, one usually finds that the Gell-Mann and Hartle mechanism for approximate decoherence works provided all the time intervals satisfy $\Delta t_k \geq \Delta t_c$, $k = 0, \ldots, N$, where Δt_c is sufficiently larger than some characteristic decoherence time scale t_D (t_D can be written in terms of σ and some parameters characterizing the environment and the system–environment couplings) [26, 27, 30]. For Δt_c one should take the smallest value compatible with a specified degree of approximate decoherence. In this sense, we can think of a coarse-graining as characterized both by the sizes σ and by the time scale Δt_c.

2.4. Effective Equations of Motion for the System

Assuming that the mechanism for approximate decoherence described in the previous subsection works, an approximate effective description of the coarse-grained system variables in terms of a set of random variables [in the sense of Eq. (2.31)] is available, at least for instants of time satisfying $\Delta t_k \geq \Delta t_c$, for $k = 0, \ldots, N$. The corresponding probability distribution $\mathcal{P}_{\bar{q}}(\{\bar{q}\})_{(t_1,\ldots,t_N)}$ is given by the diagonal elements of the decoherence function (2.22). We shall next make an estimation of this probability distribution. This follows essentially the derivation of Gell-Mann and Hartle [26, 27]. For alternative derivations for more specific models, see refs. 28, 31, and 32. Introducing the new variables $q_\Delta \equiv q_+ - q_-$ and $q_\Sigma \equiv \frac{1}{2}(q_+ + q_-)$, and similarly for $\bar{q}_{\pm k}$, and assuming that $\sigma \gg d^{(k)}$, note first, from (2.35), that the restrictions on the integration over q_Δ coming from the coarse-graining can be neglected in the diagonal elements of this decoherence function. Therefore, using (2.23) and (2.10), and writing $S_{\text{eff}}[q_+, q_-] \equiv S_{\text{eff}}[q_\Delta, q_\Sigma]$, we get

$$\mathcal{P}_{\bar{q}}(\{\bar{q}_\Sigma\})_{(t_1,\ldots,t_N)} \simeq \int \mathcal{D}[q_\Sigma] \prod_{k=1}^{N} \gamma^2(q_\Sigma(t_k) - \bar{q}_{\Sigma_k}) \mathcal{P}_{\text{f}}[q_\Sigma] \tag{2.36}$$

where

$$\mathcal{P}_{\text{f}}[q_\Sigma] \equiv \int_{q_\Delta(t_f)=0} \mathcal{D}[q_\Delta] \rho_s\left(q_{\Sigma_i} + \frac{1}{2} q_{\Delta_i}, q_{\Sigma_i} - \frac{1}{2} q_{\Delta_i}; t_i\right) \times \exp\left\{\frac{i}{h} S_{\text{eff}}[q_\Delta, q_\Sigma]\right\} \tag{2.37}$$

At this stage, we introduce two simplifications in our analysis. First, we restrict our evaluation to coarse-grained system variables having significance only up to certain scales, larger enough than σ so that the random variables $\bar{q}_k$ can be well approximated by continuous random variables. This approximation can be implemented with the use of a set of approximate projection operators $\hat{P}_{\bar{q}}(t)$, with $\bar{q}$ being continuous variables, which satisfy the properties (2.21) in an approximate way (see refs. 31 and 32 for an example). Then, all the sums $\Sigma_{\{\bar{q}\}}$ can be replaced by integrals $\int \prod_{k=1}^{N} d\bar{q}_k$ and the functions $\mathcal{P}_{\bar{q}}(\{\bar{q}\})_{(t_1,\ldots,t_N)}$ become probability densities. Second, as long as we are only interested in the dynamics of the system on time scales much larger than Δt_c (Δt_c is proportional to the decoherence time scale t_D, which is typically extremely small; see refs. 25, 39, and 43 for some examples), we can take the continuous-time limit in (2.36). In order to do so, consider the instants $t_k \equiv t_i + k\,\Delta t$, $k = 0, \ldots, N+1$, with $\Delta t \equiv (t_f - t_i)/(N+1)$. Introducing functions $\bar{q}(t)$ such that $\bar{q}(t_k) = \bar{q}_k$ (assumed now to be continuous variables), and letting $N \to \infty$ in (2.36) (replace $\bar{q}_{\Sigma_k}$ by $\bar{q}_k$), with $(t_f - t_i)$ maintained finite (thus, $\Delta t \to 0$), we get a probability distribution functional associated to some stochastic variables $\bar{q}(t)$ [5]:

$$\mathcal{P}_{\bar{q}}[\bar{q}] \simeq \int \mathcal{D}[q_\Sigma]\gamma^2[q_\Sigma - \bar{q}]\mathcal{P}_f[q_\Sigma] \tag{2.38}$$

where $\gamma[q]$ is the functional corresponding to $\prod_{k=1}^{N} \gamma(q(t_k))$ in the limit $N \to \infty$ [some redefinitions in the parameters entering in the function $\gamma(q)$ may be needed in order that such a limit is well defined; see refs. 32 and 28 for an explicit example of how this limit is taken]. Notice that, if we take the limit to the continuous in time and in the variables $\bar{q}_k$ in (2.30), we get a functional $\Phi_{\bar{q}}[j]$ which is the functional Fourier transform of $\mathcal{P}_{\bar{q}}[\bar{q}]$. Hence, $\Phi_{\bar{q}}[j]$ can be interpreted as the characteristic functional for the stochastic variables $\bar{q}(t)$ [5]. From the probability functional (2.38) or, equivalently, from the associated characteristic functional [by functional derivation with respect to the sources $j(t)$], we can compute the Green functions $G_{c\,m_1\cdots m_s}^{n_1\cdots n_r}(t'_1, \ldots, t'_r; t''_1, \ldots, t''_s)$ with each of the instants in $\{t'_1, \ldots, t'_r\}$ being separated from t_i and from the remaining instants in this set by intervals larger enough than Δt_c, and similarly for the instants in $\{t''_1, \ldots, t''_s\}$.

We can get a good approximation to the path integral (2.37) by expanding $S_{\rm eff}[q_\Delta, q_\Sigma]$ in powers of q_Δ and neglecting higher than quadratic terms, i.e., we make a Gaussian approximation in this path integral. This expansion can be made using (2.13) and writing $S_{\rm IF}^{\rm eff}[q_+, q_-] \equiv S_{\rm IF}^{\rm eff}[q_\Delta, q_\Sigma]$. In this expansion, the dependence of $S_{\rm eff}[q_\Delta, q_\Sigma]$ on the velocities $\dot{q}_\Delta(t)$ (we assume that there is no dependence on time derivatives of higher order)[6] gives rise, after integration by parts, to boundary terms proportional to q_{Δ_i} (we use that $q_{\Delta f} =$

[6] We understand that a term depends on $\dot{q}_\Delta(t)$ if it does so before any integration by parts.

0). For instance, assuming that $S_s^{\rm eff}[q] = \int dt\, L_s(q(t), \dot{q}(t), t)$, in the expansion of the terms $S_s^{\rm eff}$ we find a boundary term $-p_s(q_{\Sigma_i}, \dot{q}_{\Sigma_i}, t_i)q_{\Delta_i}$, where $p_s \equiv \partial L_s / \partial \dot{q}$ are the canonical momenta. Similarly, if $S_{\rm IF}^{\rm eff}$ depends on $\dot{q}_\Delta(t)$, its expansion will contain some boundary terms. However, since, in general, $S_{\rm IF}^{\rm eff}$ depends nonlocally on $q_\Delta(t)$ and $q_\Sigma(t)$, these terms will be more complicated. Note that we are considering models slightly more general than the ones studied by Gell-Mann and Hartle [26, 27] since we allow for the possibility of an influence action depending on $\dot{q}_\Delta(t)$ and $\dot{q}_\Sigma(t)$. The motivation for considering such a generalization is that we are interested in field theory actions with interaction terms depending on the derivatives of the fields.

One can show that, when expanding up to quadratic order in q_Δ, the general form for the boundary terms in $S_{\rm IF}^{\rm eff}$ is

$$-F_1[q_\Sigma](t_i)q_{\Delta_i} + iF_2[q_\Sigma](t_i)q_{\Delta_i}^2 + i\int dt\, q_\Delta(t)F_3[q_\Sigma](t, t_i)q_{\Delta_i}$$

where F_1, F_2, and F_3 are real functionals of q_Σ, which vanish when $S_{\rm IF}^{\rm eff}$ does not depend on $\dot{q}_\Delta(t)$. Finally, we get the following expansion:

$$\begin{aligned} S_{\rm eff}[q_\Delta, q_\Sigma] &= S_s^{\rm eff}\left[q_\Sigma + \frac{1}{2}q_\Delta\right] - S_s^{\rm eff}\left[q_\Sigma - \frac{1}{2}q_\Delta\right] + S_{\rm IF}^{\rm eff}[q_\Delta, q_\Sigma] \\ &= -p_1[q_\Sigma](t_i)q_{\Delta_i} + iF_2[q_\Sigma](t_i)q_{\Delta_i}^2 \\ &\quad + i\int dt\, q_\Delta(t)F_3[q_\Sigma](t, t_i)q_{\Delta_i} + \int dt\, q_\Delta(t)C[q_\Sigma](t) \\ &\quad + \frac{i}{2\hbar}\int dt\, dt'\, q_\Delta(t)q_\Delta(t')C_2[q_\Sigma](t, t') + O(q_\Delta^3) \end{aligned} \tag{2.39}$$

with

$$p_1[q](t_i) \equiv p_s(q_i, \dot{q}_i, t_i) + F_1[q](t_i), \qquad C[q](t) \equiv \frac{\delta S_s^{\rm eff}[q]}{\delta q(t)} + C_1[q](t) \tag{2.40}$$

and

$$C_k[q_\Sigma](t_1, \ldots, t_k) \equiv \left(\frac{\hbar}{i}\right)^{k-1} \frac{\delta^k S_{\rm IF}^{\rm eff}[q_\Delta, q_\Sigma]}{\delta q_\Delta(t_1) \cdots \delta q_\Delta, (t_k)}\Bigg|_{q_\Delta = 0} \tag{2.41}$$

where the functional derivatives with respect to $q(t)$ are defined for variations which keep the value of $q(t)$ fixed at $t = t_i$ and $t = t_f$.

Substituting the expansion (2.39) into Eq. (2.37), we get a Gaussian path integral, which can be calculated. Note that, since Im $S_{\rm IF}^{\rm eff} \geq 0$, $C_2[q](t, t')$

is positive semidefinite. In order for the Gaussian approximation that we have carried out to be valid, we must assume in addition that $C_2[q](t, t')$ is strictly positive definite and, thus, det $C_2[q] \neq 0$. We get

$$\mathcal{P}_f[q] \simeq N W_i[q][\det(C_2[q]/2\pi\hbar^2)]^{-1/2} \times \exp\left\{-\frac{1}{2}\int dt\, dt'\; C[q](t) C_2^{-1}[q](t, t') C[q](t')\right\} \tag{2.42}$$

where N is a normalization constant, C_2^{-1} is the inverse of C_2 defined by

$$\int dt''\; C_2(t, t'') C_2^{-1}(t'', t') = \delta(t - t') \tag{2.43}$$

$W_i[q] \equiv W(q(t_i), p[q](t_i), \Pi[q](t_i); t_i)$, with

$$W(q, p, \Pi; t_i) \equiv \int \frac{dq_0}{2\pi\hbar}\left[\exp\left(-\frac{i}{\hbar}\, q_0 p\right)\exp\left(-\frac{1}{\hbar}\, q_0^2\Pi\right)\right] \times \rho_s\left(q_i + \frac{1}{2}\, q_0,\, q_i - \frac{1}{2}\, q_0;\, t_i\right) \tag{2.44}$$

and

$$p[q](t_i) \equiv p_1[q](t_i) + \hbar\int dt\, dt'\; F_3[q](t, t_i) C_2^{-1}[q](t, t') C[q](t')$$

$$\Pi[q](t_i) \equiv F_2[q](t_i) - \frac{\hbar}{2}\int dt\, dt'\; F_3[q](t, t_i) C_2^{-1}[q](t, t') F_3[q](t', t_i) \tag{2.45}$$

Note that the function W defined in (2.44) is a generalization of the Wigner function associated to the initial state of the system, and it reduces to the ordinary Wigner function for $\Pi = 0$ [44]. Note that, in expression (2.42), the momenta $p[q](t_i)$ in this generalized Wigner function are in general different from the canonical momenta $p_s(q_i, \dot{q}_i, t_i)$. In the case of $S_{\rm IF}^{\rm eff}$ not depending on the velocities $\dot{q}_\Delta(t)$, one has $p[q](t_i) = p_s(q_i, \dot{q}_i, t_i)$ and $\Pi[q](t_i) = 0$, thus $W_i[q]$ is the standard Wigner function. From the definition (2.41), and using the properties of $S_{\rm IF}^{\rm eff}[q_+, q_-]$, we can see that

$$C_1[q](t) = \left.\frac{\delta\, \mathrm{Re}\, S_{\rm IF}^{\rm eff}[q_+, q_-]}{\delta q_+(t)}\right|_{q_+=q_-=q} = \left.\frac{\delta S_{\rm IF}^{\rm eff}[q_+, q_-]}{\delta q_+(t)}\right|_{q_+=q_-=q}$$

$$C_2[q](t, t') = \frac{\hbar}{2}\left[\frac{\delta^2\, \mathrm{Im}\, S_{\rm IF}^{\rm eff}[q_+, q_-]}{\delta q_+(t)\, \delta q_+(t')} - \frac{\delta^2\, \mathrm{Im}\, S_{\rm IF}^{\rm eff}[q_+, q_-]}{\delta q_+(t)\, \delta q_-(t')}\right]_{q_+=q_-=q} \tag{2.46}$$

and then, from (2.40) and (2.13), we have

$$C[q](t) = \left.\frac{\delta S_{\text{eff}}[q_+, q_-]}{\delta q_+(t)}\right|_{q_+=q_-=q} \tag{2.47}$$

Substituting (2.42) into (2.38), we see that the only nonnegligible contribution to the path integral in (2.38) comes from those paths which are not very far from the paths $q^{(0)}(t)$ which satisfy $C[q^{(0)}](t) = 0$, that is, which satisfy the semiclassical equation (2.16). This implies that only those paths $\bar{q}(t)$ which remain always near the semiclassical paths $q^{(0)}(t)$ will give a nonnegligible value to $\mathcal{P}_{\bar{q}}[\bar{q}]$. In this sense, the mechanism proposed by Gell-Mann and Hartle is a mechanism for decoherence and classicalization of coarse-grained system variables. However, we see that, in general, $\mathcal{P}_{\bar{q}}[\bar{q}]$ has a complicated functional dependence on $\bar{q}(t)$.

Let us study the deviations from a specific solution of the semiclassical equation, that is, we now restrict consideration to those paths $\bar{q}(t)$ which are distributed around a given solution $q^{(0)}(t)$ of the semiclassical equation. We can now introduce stochastic variables $\Delta q(t) \equiv \bar{q}(t) - q^{(0)}(t)$ which describe the deviations from $q^{(0)}(t)$. The associated probability distribution functional $\mathcal{P}_{\Delta q}[\Delta q]$ is equal to $\mathcal{P}_{\bar{q}}[q^{(0)} + \Delta q]$ up to a normalization factor, which, from (2.38), is given by

$$\mathcal{P}_{\bar{q}}[q^{(0)} + \Delta q] \simeq \int \mathcal{D}[q]\, \gamma^2[q]\, \mathcal{P}_f[q^{(0)} + \Delta q + q] \tag{2.48}$$

In practice, it is difficult to work out the explicit dependence of the probability distribution functional on the characteristic parameters of the coarse-graining, σ and Δt_c, even in simple models [28, 32]. Nevertheless, if such parameters are small enough so that the values of $\mathcal{P}_f[q^{(0)} + \Delta q + q]$ do not change very much for the different paths $q(t)$ which give a nonnegligible contribution in (2.48), the functional (2.48) can be approximated by $\mathcal{P}_f[q^{(0)} + \Delta q]$. We can make a further approximation by expanding $\mathcal{P}_f[q^{(0)} + \Delta q]$ around $q^{(0)}$. This can be done by setting $q_\Sigma = q^{(0)} + \Delta q$ in (2.39), expanding in Δq, and substituting the result for this expansion in (2.37). The result to lowest nontrivial order is

$$\mathcal{P}_{\Delta q}[\Delta q] \simeq N[q^{(0)}] W_i[q^{(0)} + \Delta q] \exp\left\{ -\frac{1}{2} \int dt\, dt'\, C_L[q^{(0)} + \Delta q](t) \right.$$
$$\left. \times\, C_2^{-1}[q^{(0)}](t, t') C_L[q^{(0)} + \Delta q](t') \right\} \tag{2.49}$$

where $N[q^{(0)}]$ is a normalization factor and $C_L[q^{(0)} + \Delta q]$ is the expansion of $C[q^{(0)} + \Delta q]$ to linear order in Δq. Notice that, in this probability functional, the factor $W_i[q^{(0)} + \Delta q]$ contains all the contribution arising from the initial state of the system. This generalized Wigner function, even if computed expanding around $q^{(0)}$, will have in general a complicated nonlocal dependence on Δq except when $S_{\rm IF}^{\rm eff}$ is independent of $\dot{q}_\Delta$, in which case it reduces to the standard Wigner function for the initial state of the system and depends only on Δq_i and $\Delta\dot{q}_i$. If the deviations from $q^{(0)}$ are small enough, we can approximate $W_i[q^{(0)} + \Delta q] \simeq W_i[q^{(0)}]$. Then, with these approximations, the variables Δq are distributed in such a way that $C_L[q^{(0)} + \Delta q](t)$ are Gaussian stochastic variables characterized by

$$\langle C_L[q^{(0)} + \Delta q](t)\rangle_c = 0,$$
$$\langle C_L[q^{(0)} + \Delta q](t) C_L[q^{(0)} + \Delta q](t')\rangle_c = C_2[q^{(0)}](t, t') \tag{2.50}$$

Thus, the equation of motion for Δq is the Langevin equation

$$C_L[q^{(0)} + \Delta q](t) + \xi(t) = 0 \tag{2.51}$$

where $\xi(t)$ is a Gaussian stochastic source with

$$\langle \xi(t)\rangle_c = 0, \qquad \langle \xi(t)\xi(t')\rangle_c = C_2[q^{(0)}](t, t') \tag{2.52}$$

We should mention that there are very simple models for quantum Brownian motion in which all the actions involved are quadratic in their variables and the interaction terms are independent of the velocities [4, 5, 9, 10, 26–28, 31, 37, 40]. For such models, assuming that the environment is in an initial state of thermal equilibrium, the influence functional can be computed exactly and it is Gaussian. The effective action of Feynman and Vernon in these cases is exactly of the form (2.39), with $C_1[q_\Sigma](t)$ linear in q_Σ, $C_2(t, t')$ independent of q_Σ, and $F_1 = F_2 = F_3 = 0$. Thus, for these models, expression (2.42) is actually exact. In these cases, with the approximation $\mathcal{P}_{\bar{q}}[\bar{q}] \simeq \mathcal{P}_f[\bar{q}]$, one can derive a Langevin equation for the stochastic variables $\bar{q}(t)$ without the need of introducing a specific solution $q^{(0)}$ of the semiclassical equation. This Langevin equation is simply $C[\bar{q}](t) + \xi(t) = 0$, with $\xi(t)$ a Gaussian stochastic source with $\langle\xi(t)\rangle_c = 0$ and $\langle\xi(t)\,\xi(t')\rangle_c = C_2(t, t')$. However, for models with more complicated actions, we are only able to derive effective equations of motion for the deviations Δq around a given solution $q^{(0)}$ of the semiclassical equation.

2.5. A Quick Method to Obtain the Langevin Equation

Starting with the effective action of Feynman and Vernon (2.13), there is a quick way to obtain the Langevin equation (2.51) for the deviations Δq around a specific solution of the semiclassical equation. This method has actually been extensively used in the literature in the context of quantum Brownian motion [9, 11, 17] and also in the context of field theory [12, 18–21], including some models for gravity interacting with a scalar field

[6–8, 13–16]. One starts with an expansion of this effective action around a solution $q^{(0)}(t)$ of the semiclassical equation up to quadratic order in perturbations $\Delta q_\pm$ satisfying $\Delta q_+(t_i) = \Delta q_-(t_i)$ and $\Delta q_+(t_f) = \Delta q_-(t_f)$ (in the simplest models, in which this effective action is exactly quadratic in q_+ and q_-, one works directly with the exact expression). From (2.39), it is easy to see that the expansion for the influence action reads

$$
\begin{aligned}
&S^{\text{eff}}_{\text{IF}}[q^{(0)} + \Delta q_+, q^{(0)} + \Delta q_-] \\
&\quad = \int dt(\Delta q_+(t) - \Delta q_-(t))C_1\left[q^{(0)} + \frac{1}{2}(\Delta q_+ + \Delta q_-)\right](t) \\
&\qquad + \frac{i}{2\hbar}\int dt\, dt'(\Delta q_+(t) - \Delta q_-(t))C_2[q^{(0)}](t,t')(\Delta q_+(t') \\
&\qquad - \Delta q_-(t')) + O(\Delta q^3)
\end{aligned}
\tag{2.53}
$$

where it is understood that C_1 has to be expanded up to linear order. Using the identity, which follows from a Gaussian path integration,

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2\hbar^2}\int dt\, dt'(\Delta q_+(t) - \Delta q_-(t))C_2[q^{(0)}](t,t')(\Delta q_+(t') - \Delta q_-(t'))\right\} \\
&\quad = \int \mathcal{D}[\xi]\mathcal{P}_\xi[\xi]\exp\left\{\frac{i}{\hbar}\int dt\, \xi(t)(\Delta q_+(t) - \Delta q_-(t))\right\}
\end{aligned}
\tag{2.54}
$$

where $\mathcal{P}_\xi[\xi]$ is the Gaussian probability distribution functional for the Gaussian stochastic variables $\xi(t)$ characterized by (2.52), that is,

$$
\begin{aligned}
&\mathcal{P}_\xi[\xi]\,\frac{\exp\left\{-\frac{1}{2}\int dt\, dt'\xi(t)C_2^{-1}[q^{(0)}](t,t')\xi(t')\right\}}{M} \\
&M \equiv \mathcal{D}[\xi]\exp - \left\{\frac{1}{2}\int d\tau\, d\tau'\xi(\tau)C_2^{-1}[q^{(0)}](\tau,\tau')\xi(\tau')\right\}
\end{aligned}
\tag{2.55}
$$

we can write in this approximation

$$
\begin{aligned}
&\left|\mathcal{F}^{\text{eff}}_{\text{IF}}[q^{(0)} + \Delta q_+, q^{(0)} + \Delta q_-]\right| \\
&\quad = \exp\left\{-\frac{1}{\hbar}\,\text{Im}\, S^{\text{eff}}_{\text{IF}}[q^{(0)} + \Delta q_+, q^{(0)} + \Delta q_-]\right\}
\end{aligned}
$$

$$= \left\langle \exp\left\{ \frac{i}{\hbar} \int dt\, \xi(t)(\Delta q_+(t) - \Delta q_-(t)) \right\} \right\rangle_{\xi} \qquad (2.56)$$

where $\langle\cdot\rangle_c$ means statistical average over the stochastic variables $\xi(t)$. Thus, the effect of the imaginary part of the influence action (2.53) on the corresponding influence functional is equivalent to the averaged effect of the stochastic source $\xi(t)$ coupled linearly to the perturbations $\Delta q_\pm$ (note that in the above expressions the perturbations $\Delta q_\pm$ are deterministic functions). Notice that expression (2.54) or, equivalently, (2.56) gives the characteristic functional of the stochastic variables $\xi(t)$ [5]. The influence functional in the approximation (2.53) can then be written as an statistical average over ξ:

$$\mathscr{F}_{\rm IF}^{\rm eff}[q^{(0)} + \Delta q_+, q^{(0)} + \Delta q_-]$$
$$= \left\langle \exp\left\{ \frac{i}{\hbar} \mathscr{A}_{\rm IF}^{\rm eff}[\Delta q_+, \Delta q_-; \xi] \right\} \right\rangle_{\xi} \qquad (2.57)$$

where

$$\mathscr{A}_{\rm IF}^{\rm eff}[\Delta q_+, \Delta q_-; \xi] \equiv {\rm Re} S_{\rm IF}^{\rm eff}[q^{(0)} + \Delta q_+, q^{(0)} + \Delta q_-]$$
$$+ \int dt\, \xi(t)(\Delta q_+(t) - \Delta q_-(t)) + O(\Delta q^3) \quad (2.58)$$

where ${\rm Re}\, S_{\rm IF}^{\rm eff}$ can be read from expression (2.53). The Langevin equation (2.51) can be easily derived from the action

$$\mathscr{A}_{\rm eff}[\Delta q_+, \Delta q_-; \xi] \equiv S_s^{\rm eff}[q^{(0)} + \Delta q_+] - S_s^{\rm eff}[q^{(0)} + \Delta q_-]$$
$$+ \mathscr{A}_{\rm IF}^{\rm eff}[\Delta q_+, \Delta q_-; \xi] \qquad (2.59)$$

where $S_s^{\rm eff}[q^{(0)} + \Delta q_\pm]$ has to be expanded up to second order in the perturbations $\Delta q_\pm$. That is,

$$\left. \frac{\delta \mathscr{A}_{\rm eff}[\Delta q_+, \Delta q_-; \xi]}{\delta \Delta q_+(t)} \right|_{\Delta q_+ = \Delta q_- = \Delta q} = 0 \qquad (2.60)$$

leads to Eq. (2.51).

3. EFFECTIVE EQUATIONS OF MOTION FOR THE GRAVITATIONAL FIELD

In this section, we shall apply the results of the previous section to derive effective equations of motion for the gravitational field in a semiclassical regime. In order to do so, we will consider the simplest case of a linear real scalar field Φ coupled to the gravitational field. We restrict ourselves to the case of fields defined on a globally hyperbolic manifold $\mathcal{M}$.

In this case, we consider the metric field $g_{ab}(x)$ as the system degrees of freedom and the scalar field $\Phi(x)$ and also some "high-momentum" gravitational modes, considered as inaccessible to the observations, as the environment variables. Unfortunately, since the form of a complete quantum theory of gravity interacting with matter is unknown, we do not know what these "high-momentum" gravitational modes are. Such a fundamental quantum theory might not even be a field theory, in which case the metric and scalar fields would not be fundamental objects. Thus, in this case, we cannot attempt to evaluate the effective actions in Eq. (2.11) starting from the fundamental quantum theory and integrating out the "high-momentum" gravitational modes. What we can do instead is to adopt the usual procedure when dealing with an effective quantum field theory. That is, we shall take for the actions $S_s^{\text{eff}}[g]$ and $S_{se}^{\text{eff}}[g^+, \Phi_+; g^-, \Phi_-]$ the most general local form compatible with general covariance and with the properties of S_{se}^{eff} [these properties are analogous to those of S_{IF} in Eq. (2.8)] [35, 45]. The general form for $S_s^{\text{eff}}[g]$ is

$$S_s^{\text{eff}}[g] = \int d^4x \ \sqrt{-g}\left[\frac{1}{16\pi G_B}(R - 2\Lambda_B) + \alpha_B C_{abcd}C^{abcd} + \beta_B R^2 + \dots\right] \tag{3.1}$$

where R and C_{abcd} are, respectively, the scalar curvature and the Weyl tensor associated to the metric g_{ab}, $1/G_B$, Λ_B/G_B, α_B, and β_B are bare coupling constants, and the dots represent terms of higher order in the curvature [because of the Gauss–Bonnet theorem in four spacetime dimensions, there is no need to consider terms of second order in the curvature different from those written in Eq. (3.1)]. Since $\mathcal{M}$ is a globally hyperbolic manifold, we can foliate it by a family of Cauchy hypersurfaces Σ_t, labeled by a time coordinate t. We use the notation $\mathbf{x}$ for spatial coordinates on each of these hypersurfaces, and t_i and t_f for some initial and final times, respectively. The integration domain for all the action terms must now be understood as a compact region $\mathcal{U}$ of the manifold $\mathcal{M}$, bounded by the hypersurfaces Σ_{t_i} and Σ_{t_f} (i.e., as in the previous section, integrals in t are integrals between t_i and t_f).

For the matter part of the effective action, let us consider the following ansatz:

$$S_{se}^{\text{eff}}[g^+, \Phi_+; g^-, \Phi_-] = S_m[g^+, \Phi_+] - S_m[g^-, \Phi_-] \tag{3.2}$$

with

$$S_m[g, \Phi] \equiv -\frac{1}{2}\int d^4x \ \sqrt{-g}[g^{ab}\partial_a\Phi\partial_b\Phi + (m^2 + \xi R)\Phi^2 + \cdots] \tag{3.3}$$

where ξ is a dimensionless coupling parameter of the field to the scalar curvature, and the dots stand for terms of higher order in the curvature and in the number of derivatives of the scalar field. Self-interaction terms for the scalar field could also be included, but, for simplicity, we shall ignore them in this paper. One can see that general covariance and the properties of $S_{se}^{\rm eff}[g^+, \Phi_+; g^-, \Phi_-]$ imply that imaginary terms and terms mixing the "plus" and "minus" fields in this action must be necessarily nonlocal. Thus, within a local approximation, the ansatz (3.2) is the most general form for this action. We shall comment below on some limitations of this local approximation.

In order to simplify the analysis, we neglect the contributions of the higher order terms not written in Eqs. (3.1) and (3.3). Assuming that the mass of the scalar field is much smaller than the Planck mass, this is a good approximation in a regime where all the characteristic curvature scales are far enough from the Planck scales. The terms in the gravitational Lagrangian density proportional to R^2 and $C_{abcd}C^{abcd}$ need to be considered in order to renormalize the matter one-loop ultraviolet divergences.

Assuming the form (3.2) for the matter part of the effective action, we can now introduce the corresponding effective influence functional as in Eq. (2.12). Let us assume that the state of the scalar field in the Schrödinger picture at the initial time $t = t_i$ is described by a density operator $\hat{\rho}^{\rm s}(t_i)$ [in the notation of the previous section, this was $\hat{\rho}_e^{\rm s}(t_i)$, but here we drop the index e to simplify the notation]. If we now consider the theory of a scalar field quantized in a classical background spacetime $(\mathcal{M}, g_{ab})$ through the action (3.3), to this state there would correspond a state in the Heisenberg picture described by a density operator $\hat{\rho}[g]$. Let $\{|\varphi(\mathbf{x})\rangle^{\rm s}\}$ be the basis of eigenstates of the Schrödinger-picture scalar field operator $\hat{\Phi}^{\rm s}(\mathbf{x})$: $\hat{\Phi}^{\rm s}(\mathbf{x})|\varphi\rangle^{\rm s} = \varphi(\mathbf{x})|\varphi\rangle^{\rm s}$. The matrix elements of $\hat{\rho}^{\rm s}(t_i)$ in this basis will be written as $\rho_i[\varphi, \tilde{\varphi}] \equiv {}^{\rm s}\langle\varphi|\hat{\rho}^{\rm s}(t_i)|\tilde{\varphi}\rangle^{\rm s}$. We can now introduce the effective influence functional as

$$\mathcal{F}_{\rm IF}^{\rm eff}[g^+, g^-] \equiv \int \mathcal{D}[\Phi_+]\mathcal{D}[\Phi_-]\rho_i[\Phi_+(t_i), \Phi_-(t_i)] \times \delta[\Phi_+(t_f) - \Phi_-(t_f)]\, e^{i(S_m[g^+,\Phi_+]-S_m[g^-,\Phi_-])} \tag{3.4}$$

and the effective influence action will be given by

$$\mathcal{F}_{\rm IF}^{\rm eff}[g^+, g^-] \equiv e^{iS_{\rm IF}^{\rm eff}[g^+,g^-]}$$

Of course, trying to show how the mechanism for decoherence and classicalization of the previous section can work in this case would involve some technical difficulties, such as introducing diffeomorphism-invariant coarse-grainings and eliminating properly the gauge redundancy (with the use of some suitable Faddeev–Popov method) in the path integrals. We are not going to deal with such issues in this paper. We rather assume that they can be suitably implemented without changing the main results for the effective equations of motion.

Expression (3.4) is actually formal; it is ill defined and must be regularized in order to get a meaningful quantity for the influence functional. We shall formally assume that we can regularize it using dimensional regularization, that is, that we can give sense to Eq. (3.4) by dimensional continuation of all the quantities that appear in this expression. We should mention that, however, when performing specific calculations, the dimensional regularization procedure may not be the most suitable one in all cases. In this sense, one should understand the following derivation as being formal. Using dimensional regularization, we must substitute the action S_m in (3.4) by some generalization to n spacetime dimensions. This can be taken as

$$S_m[g, \Phi_n] = -\frac{1}{2}\int d^n x \; \sqrt{-g}[g^{ab}\partial_a\Phi_n\partial_b\Phi_n + (m^2 + \xi R)\Phi_n^2] \tag{3.5}$$

where we use a notation in which we write an index n in all the quantities that have different physical dimensions than the corresponding physical quantities in the spacetime of four dimensions. The quantities that do not carry an index n have the same physical dimensions as the corresponding ones in four spacetime dimensions, although they should not be confused with such physical quantities. A quantity with an index n can always be associated to another one without an index n; these are related by some mass scale μ; for instance, it is easy to see that $\Phi_n = \mu^{(n-4)/2}\Phi$.

In order to write the effective equations for the metric field in dimensional regularization, we need to substitute the action (3.1) by some suitable generalization to n spacetime dimensions. We take

$$S_s^{\text{eff}}[g] = \mu^{n-4}\int d^m x \; \sqrt{-g}\left[\frac{1}{16\pi G_B}(R - 2\Lambda_B) + \frac{2}{3}\alpha_B(R_{abcd}R^{abcd} - R_{ab}R^{ab}) + \beta_B R^2\right] \tag{3.6}$$

where R_{abcd} is the Riemann tensor and, again, the mass parameter μ has been introduced in order to get the correct physical dimensions. Using the Gauss–Bonnet theorem in four spacetime dimensions, one can see that the action obtained by setting $n = 4$ in (3.6) is equivalent to (3.1). The form of the action (3.6) is suggested from the Schwinger–DeWitt analysis of the divergencies in the stress-energy tensor in dimensional regularization [46]. The effective action of Feynman and Vernon (2.13) is in our case given by $S_{\text{eff}}[g^+, g^-] = S_s^{\text{eff}}[g^+] - S_s^{\text{eff}}[g^-] + S_{\text{IF}}^{\text{eff}}[g^+, g^-]$. Since the action terms (3.5) and (3.6) contain second-order derivatives of the metric, one should also add some boundary terms to them [33, 7]. The effect of these boundary terms is simply to cancel out the boundary terms that appear when taking variations of $S_{\rm eff}[g^+, g^-]$ that keep the values of g^+_{ab} and g^-_{ab} fixed on the boundary of $\mathcal{U}$. They guarantee that we can obtain an expansion for $S_{\rm eff}[g^+, g^-]$ analogous to (2.39), with no extra boundary terms coming from the integration by parts of terms containing second-order derivatives of $g^\Delta_{ab} \equiv g^+_{ab} - g^-_{ab}$. Alternatively, in order to obtain the effective equations for the metric [equations analogous to (2.16) and (2.51)], we can work with the action terms (3.5) and (3.6) (without boundary terms) and neglect all boundary terms when taking variations with respect to $g^\pm_{ab}$. From now on, all the functional derivatives with respect to the metric must be understood in this sense.

3.1. The Semiclassical Einstein Equation

From the action (3.5), we can define the stress-energy tensor functional in the usual way,

$$T^{ab}[g, \Phi_n](x) \equiv \frac{2}{\sqrt{-g(x)}} \frac{\delta S_m[g, \Phi_n]}{\delta g_{ab}(x)} \tag{3.7}$$

which yields

$$\begin{aligned} T^{ab}[g, \Phi_n] = {} & \nabla^a \Phi_n \nabla^b \Phi_n - \tfrac{1}{2} g^{ab} \nabla^c \Phi_n \nabla_c \Phi_n \\ & - \tfrac{1}{2} g^{ab} m^2 \Phi_n^2 + \xi (g^{ab} \Box - \nabla^a \nabla^b + G^{ab}) \Phi_n^2 \end{aligned} \tag{3.8}$$

where ∇_a is the covariant derivative associated to the metric g_{ab}, $\Box \equiv \nabla_a \nabla^a$, and G_{ab} is the Einstein tensor. Working in the Heisenberg picture, we can now formally introduce the stress-energy tensor operator for a scalar field quantized in a classical spacetime background, regularized using dimensional regularization, as

$$\hat{T}_n^{ab}[g] \equiv T^{ab}[g, \hat{\Phi}_n[g]], \qquad \hat{T}^{ab}[g] \equiv \mu^{-(n-4)} \hat{T}_n^{ab}[g] \tag{3.9}$$

where $\hat{\Phi}_n[g](x)$ is the Heisenberg-picture field operator in n spacetime dimensions, which satisfies the Klein–Gordon equation

$$(\Box - m^2 - \xi R)\hat{\Phi}_n = 0 \tag{3.10}$$

and where we use a symmetrical ordering (Weyl ordering) prescription for the operators. Using Eq. (3.10), one can write the stress-energy operator in the following way:

$$\hat{T}_n^{ab}[g] = \tfrac{1}{2}\{\nabla^a \hat{\Phi}_n[g], \nabla^b \hat{\Phi}_n[g]\} + \mathcal{D}^{ab}[g] \hat{\Phi}_n^2[g] \tag{3.11}$$

where $\mathcal{D}^{ab}[g]$ is the differential operator

$$\mathcal{D}_x^{ab} \equiv (\xi - \tfrac{1}{4})g^{ab}(x)\Box_x + \xi(R^{ab}(x) - \nabla_x^a \nabla_x^b) \tag{3.12}$$

with R_{ab} the Ricci tensor. From the definitions (3.4), (3.7), and (3.9), one can see that

$$\frac{2}{\sqrt{-g(x)}} \frac{\delta S_{\rm IF}^{\rm eff}[g^+, g^-]}{\delta g_{ab}^+(x)}\bigg|_{g^+=g^-=g} = \langle \hat{T}_n^{ab}(x)\rangle[g] \tag{3.13}$$

where the expectation value is taken in the n-dimensional spacetime generalization of the state described by $\hat{\rho}[g]$.

As in Eq. (2.16), if we derive $S_{\rm eff}[g^+, g^-]$ with respect to g_{ab}^+ and then set $g_{ab}^+ = g_{ab}^- = g_{ab}$, we get the semiclassical Einstein equation in dimensional regularization:

$$\frac{1}{8\pi G_B}(G^{ab}[g] + \Lambda_B g^{ab}) - \left(\frac{4}{3}\alpha_B D^{ab} + 2\beta_B B^{ab}\right)[g] = \mu^{-(n-4)}\langle \hat{T}_n^{ab}\rangle[g] \tag{3.14}$$

where the tensors D^{ab} and B^{ab} are defined as

$$\begin{aligned} D^{ab} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^n x \ \sqrt{-g}(R_{cdef}R^{cdef} - R_{cd}R^{cd}) \\ &= \frac{1}{2} g^{ab}(R_{cdef}R^{cdef} - R_{cd}R^{cd} + \Box R) \\ &\quad - 2R^{acde}R^b_{\ cde} - 2R^{acbd}R_{cd} + 4R^{ac}R_c^b - 3\Box R^{ab} + \nabla^a\nabla^b R \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} B^{ab} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^n x \ \sqrt{-g}R^2 \\ &= \frac{1}{2} g^{ab}R^2 - 2RR^{ab} + 2\nabla^a\nabla^b R - 2g^{ab}\Box R \end{aligned} \tag{3.16}$$

From equation (3.14), after renormalizing the coupling constants in order to eliminate the divergencies in $\mu^{-(n-4)}\langle \hat{T}_n^{ab}\rangle[g]$ in the limit $n \to 4$ and then taking this limit, we will get the semiclassical Einstein equation in the physical spacetime of four dimensions:

$$\frac{1}{8\pi G}(G^{ab}[g] + \Lambda g^{ab}) - 2(\alpha A^{ab} + \beta B^{ab})[g] = \langle \hat{T}_R^{ab}\rangle[g] \tag{3.17}$$

In the last equation $1/G$, Λ/G, α, and β are renormalized coupling constants, $\langle \hat{T}_R^{ab}\rangle[g]$ is the renormalized expectation value of the stress-energy tensor operator, and we have used that, for $n = 4$, $D^{ab} = (3/2)A^{ab}$, with A^{ab} the local curvature tensor obtained by functional derivation with respect to the metric of the action term corresponding to the Lagrangian density $C_{abcd}C^{abcd}$.

3.2. The Semiclassical Einstein–Langevin Equation

According to the results of the previous section, assuming that some suitably coarse-grained metric field satisfies the conditions for approximate decoherence and that the approximations of Section 2.4 are valid in a certain regime, small deviations from a given solution g_{ab} of the semiclassical Einstein equation (3.17) can be described by linear stochastic perturbations h_{ab} to that semiclassical metric. These perturbations satisfy a Langevin equation of the form (2.51), which shall be called the semiclassical Einstein–Langevin equation. Our next step will be to write the semiclassical Einstein–Langevin equation in dimensional regularization. Let us assume that g_{ab} is a solution of Eq. (3.14) in n spacetime dimensions. The semiclassical Einstein–Langevin equation in dimensional regularization then has the form

$$\frac{1}{8\pi G_B}\left(G_L^{ab}[g+h]+\Lambda_B(g^{ab}-h^{ab})\right)-\left(\frac{4}{3}\alpha_B D_L^{ab}+2\beta_B B_L^{ab}\right)[g+h]$$
$$=\mu^{-(n-4)}\langle\hat{T}_n^{ab}\rangle_L[g+h]+2\mu^{-(n-4)}\xi_n^{ab} \tag{3.18}$$

where h_{ab} is a linear stochastic perturbation to g_{ab}, $h^{ab} \equiv g^{ac}g^{bd}h_{cd}$, that is, $g^{ab} - h^{ab} + O(h^2)$ is the inverse of the metric $g_{ab} + h_{ab}$, and, as in the previous section, we use an index L to denote an expansion up to linear order in h_{ab}. In this equation, $\langle\hat{T}_n^{ab}\rangle[g+h]$ is the expectation value of $\hat{T}_n^{ab}[g+h]$ in the n-dimensional spacetime generalization of the state described by $\hat{\rho}[g+h]$, and ξ_n^{ab} is a Gaussian stochastic tensor characterized by the correlators

$$\langle\xi_n^{ab}(x)\rangle_c = 0, \qquad \langle\xi_n^{ab}(x)\xi_n^{cd}(y)\rangle_c = N_n^{abcd}[g](x,y) \tag{3.19}$$

with [see Eqs. (2.52) and (2.46)]

$$2N_n^{abcd}[g](x,y) \equiv \frac{1}{\sqrt{-g(x)}\,\sqrt{-g(y)}}\left[\frac{\delta^2\,\mathrm{Im}\,S_{\mathrm{IF}}^{\mathrm{eff}}[g^+,g^-]}{\delta g_{ab}^+(x)\,\delta g_{cd}^+(y)} - \frac{\delta^2\,\mathrm{Im}\,S_{\mathrm{IF}}^{\mathrm{eff}}[g^+,g^-]}{\delta g_{ab}^+(x)\,\delta g_{cd}^-(y)}\right]\Bigg|_{g^+=g^-=g} \tag{3.20}$$

We can write Eq. (3.18) in a more explicit way by working out the expansion $\langle\hat{T}_n^{ab}\rangle_L[g+h]$. Since, from Eq. (3.13), we have that

$$\langle\hat{T}_n^{ab}(x)\rangle[g+h]$$
$$=\frac{2}{\sqrt{-\det(g+h)(x)}}\frac{\delta S_{\mathrm{IF}}^{\mathrm{eff}}[g+h^+,g+h^-]}{\delta h_{ab}^+(x)}\Bigg|_{h^+=h^-=h} \tag{3.21}$$

this expansion can be obtained from an expansion of the influence action $S_{\rm IF}^{\rm eff}[g + h^+, g + h^-]$ up to second order in $h_{ab}^{\pm}$ (in this expansion, we can neglect boundary terms). At the same time, we can obtain a more explicit expression for the noise kernel (3.20). To perform this expansion for the influence action, we have to compute the first- and second-order functional derivatives of $S_{\rm IF}^{\rm eff}[g^+, g^-]$ and then set $g_{ab}^+ = g_{ab}^- = g_{ab}$. If we do so using the path integral representation (3.4), we can interpret these derivatives as expectation values of operators in the Heisenberg picture for a scalar field quantized in a classical spacetime background $(\mathcal{M}, g_{ab})$ as, for instance, in expression (3.13). The relevant second-order derivatives are

$$
\begin{aligned}
&\frac{1}{\sqrt{-g(x)}\,\sqrt{-g(y)}}\,\frac{\delta^2 S_{\rm IF}^{\rm eff}[g^+, g^-]}{\delta g_{ab}^+(x)\,\delta g_{cd}^+(y)}\bigg|_{g^+=g^-=g}\\
&\quad= -H_{{\rm S}_n}^{abcd}[g](x, y) - K_n^{abcd}[g](x, y) + iN_n^{abcd}[g](x, y),\\
&\frac{1}{\sqrt{-g(x)}\,\sqrt{-g(y)}}\,\frac{\delta^2 S_{\rm IF}^{\rm eff}[g^+, g^-]}{\delta g_{ab}^+(x)\,\delta g_{cd}^-(y)}\bigg|_{g^+=g^-=g}\\
&\quad= -H_{{\rm A}_n}^{abcd}[g](x, y) - iN_n^{abcd}[g](x, y)
\end{aligned}
\tag{3.22}
$$

with

$$
\begin{aligned}
N_n^{abcd}[g](x, y) &= \frac{1}{8}\,\langle\{\hat{T}_n^{ab}(x) - \langle\hat{T}_n^{ab}(x)\rangle, \hat{T}_n^{cd}(y) - \langle\hat{T}_n^{cd}(y)\rangle\}\rangle[g]\\
H_{{\rm S}_n}^{abcd}[g](x, y) &= \frac{1}{4}\,{\rm Im}\,\langle {\rm T}^*(\hat{T}_n^{ab}(x)\hat{T}_n^{cd}(y))\rangle[g]\\
H_{{\rm A}_n}^{abcd}[g](x, y) &= -\frac{i}{4}\left\langle \frac{1}{2}\,[\hat{T}_n^{ab}(x), \hat{T}_n^{cd}(y)]\right\rangle[g]\\
K_n^{abcd}[g](x, y) &= \frac{-1}{\sqrt{-g(x)}\,\sqrt{-g(y)}}\left\langle \frac{\delta^2 S_m[g, \Phi_n]}{\delta g_{ab}(x)\,\delta g_{cd}(y)}\bigg|_{\Phi_n=\hat{\Phi}_n}\right\rangle[g]
\end{aligned}
\tag{3.23}
$$

using again a symmetrical ordering (Weyl ordering) prescription for the operators in the last of these expressions. All the expectation values in these expressions are in the n-dimensional spacetime generalization of the state described by $\hat{\rho}[g]$. In the above equations, $\{\cdot, \cdot\}$ and $[\cdot, \cdot]$ mean, respectively, the anticommutator and the commutator, and we use the symbol T* to denote that, first, we have to time order the field operators $\hat{\Phi}_n$ and then apply the derivative operators that appear in each term of the product $T^{ab}(x)T^{cd}(y)$, where T^{ab} is the functional (3.8). For instance,

$$\mathrm{T}^*(\nabla^a_x \hat{\Phi}_n(x) \nabla^b_x \hat{\Phi}_n(x) \nabla^c_y \hat{\Phi}_n(y) \nabla^d_y \hat{\Phi}_n(y)) = \lim_{\substack{x_1, x_2 \to x \\ x_3, x_4 \to y}} \nabla^a_{x_1} \nabla^b_{x_2} \nabla^c_{x_3} \nabla^d_{x_4} \mathrm{T}(\hat{\Phi}_n(x_1)\hat{\Phi}_n(x_2)\hat{\Phi}_n(x_3)\hat{\Phi}_n(x_4)), \tag{3.24}$$

where T is the usual time ordering. Notice that all the kernels that appear in expressions (3.22), are real.

In fact, from (3.23) we see that the noise kernel $N^{abcd}_{\mathrm{A}_n}$ and also the kernel $H^{abcd}_{A_n}$, are free of ultraviolet divergences in the limit $n \to 4$. This is because, for a linear quantum field, the ultraviolet divergencies in $\langle \hat{T}^{ab}_n(x) \hat{T}^{cd}_n(y) \rangle$ are the same as those of $\langle \hat{T}^{ab}_n(x) \rangle \langle \hat{T}^{cd}_n(y) \rangle$. Therefore, in the semiclassical Einstein–Langevin equation (3.18), one can perform exactly the same renormalization procedure as the one for the semiclassical Einstein equation (3.14). After this renormalization procedure, Eq. (3.18) will yield the semiclassical Einstein–Langevin equation in the physical spacetime ($n = 4$). It can be written as

$$\frac{1}{8\pi G}(G^{ab}_L[g+h] + \Lambda(g^{ab} - h^{ab})) - 2(\alpha A^{ab}_L + \beta B^{ab}_L)[g+h] = \langle \hat{T}^{ab}_R \rangle_L [g+h] + 2\xi^{ab} \tag{3.25}$$

where ξ^{ab} is a Gaussian stochastic tensor with

$$\langle \xi^{ab}(x) \rangle_c = 0, \qquad \langle \xi^{ab}(x) \xi^{cd}(y) \rangle_c = N^{abcd}[g](x, y) \tag{3.26}$$

where $N^{abcd} \equiv \lim_{n \to 4} \mu^{-2(n-4)} N^{abcd}_n$. Notice from (3.23) that the noise kernel $N^{abcd}[g](x, y)$ gives a measure of the lowest order fluctuations of the scalar field stress-energy tensor around its expectation value. Thus, the stochastic metric perturbations h_{ab}, solution of the semiclassical Einstein–Langevin equation (3.25), account for the backreaction of such matter stress-energy fluctuations on the spacetime geometry. For a more detailed analysis of the semiclassical Einstein–Langevin equation and some of its applications, see ref. 47.

Going back to the expressions in dimensional regularization, which may be useful for calculational purposes, we can now write the expansion of the influence action around a given metric g_{ab}. From (3.13) and (3.22), taking into account that $S^{\mathrm{eff}}_{\mathrm{IF}}[g, g] = 0$ and that $S^{\mathrm{eff}}_{\mathrm{IF}}[g^-, g^+] = -S^{\mathrm{eff}*}_{\mathrm{IF}}[g^+, g^-]$, we get

$$S^{\mathrm{eff}}_{\mathrm{IF}}[g + h^+, g + h^-] = \frac{1}{2} \int d^n x \sqrt{-g(x)} \langle \hat{T}^{ab}_n(x) \rangle [g](h^+_{ab}(x) - h^-_{ab}(x)) - \frac{1}{2} \int d^n x \, d^n y \sqrt{-g(x)} \sqrt{-g(y)} (H^{abcd}_{\mathrm{S}_n}[g](x, y)$$

$$+ K_n^{abcd}[g](x, y))(h_{ab}^+(x)h_{cd}^+(y) - h_{ab}^-(x)h_{cd}^-(y))$$

$$- \frac{1}{2}\int d^nx\, d^ny\, \sqrt{-g(x)}\, \sqrt{-g(y)} H_{A_n}^{abcd}[g](x, y)(h_{ab}^+(x)h_{cd}^-(y)$$

$$- h_{ab}^-(x)h_{cd}^+(y))$$

$$+ \frac{i}{2}\int d^nx\, d^ny\, \sqrt{-g(x)}\, \sqrt{-g(y)} N_n^{abcd}[g](x, y)(h_{ab}^+(x)$$

$$- h_{ab}^-(x))(h_{cd}^+(y) - h_{cd}^-(y)) + O(h^3) \tag{3.27}$$

From (3.23), it is easy to see that the kernels satisfy the symmetry relations

$$H_{S_n}^{abcd}(x, y) = H_{S_n}^{cdab}(y, x), \qquad H_{A_n}^{abcd}(x, y) = -H_{A_n}^{cdab}(y, x), \tag{3.28}$$

$$K_n^{abcd}(x, y) = K_n^{cdab}(y, x)$$

Using these relations and defining

$$H_n^{abcd}(x, y) \equiv H_{S_n}^{abcd}(x, y) + H_{A_n}^{abcd}(x, y) \tag{3.29}$$

we can write the expansion (3.27) as

$$S_{\mathrm{IF}}^{\mathrm{eff}}[g + h^+, g + h^-]$$

$$= \frac{1}{2}\int d^nx\, \sqrt{-g(x)} \langle \hat{T}_n^{ab}(x)\rangle[g][h_{ab}(x)]$$

$$- \frac{1}{2}\int d^nx\, d^ny\, \sqrt{-g(x)}\, \sqrt{-g(y)}[h_{ab}(x)](H_n^{abcd}[g](x, y)$$

$$+ K_n^{abcd}[g](x, y))\{h_{cd}(y)\}$$

$$+ \frac{i}{2}\int d^nx\, d^ny\, \sqrt{-g(x)}\, \sqrt{-g(y)}[h_{ab}(x)]N_n^{abcd}[g](x, y)[h_{cd}(y)]$$

$$+ O(h^3) \tag{3.30}$$

where we have used the notation

$$[h_{ab}] \equiv h_{ab}^+ - h_{ab}^-, \qquad \{h_{ab}\} \equiv h_{ab}^+ + h_{ab}^- \tag{3.31}$$

Using this expansion and noting, from (3.23), that

$$K_n^{abcd}[g](x, y) = -\frac{1}{4}\langle \hat{T}_n^{ab}(x)\rangle[g]\, \frac{g^{cd}(x)}{\sqrt{-g(y)}}\, \delta^n(x - y)$$

$$- \frac{1}{2}\frac{1}{\sqrt{-g(y)}}\left\langle \frac{\delta T^{ab}[g, \Phi_n](x)}{\delta g_{cd}(y)}\bigg|_{\Phi_n=\hat{\Phi}_n}\right\rangle[g] \tag{3.32}$$

we get, from (3.21),

$$\langle \hat{T}_n^{ab}(x)\rangle_L[g+h] = \langle \hat{T}_n^{ab}(x)\rangle[g] + \langle \hat{T}_n^{(1)ab}[g;h](x)\rangle[g] - 2\int d^ny\ \sqrt{-g(y)}\ H_n^{abcd}[g](x,y)h_{cd}(y) \quad (3.33)$$

where the operator $\hat{T}_n^{(1)ab}$ is defined from the term of first order in the expansion $T_L^{ab}[g+h, \Phi_n]$ as

$$T_L^{ab}[g+h,\Phi_n] = T^{ab}[g,\Phi_n] + T^{(1)ab}[g,\Phi_n;h], \quad (3.34)$$
$$\hat{T}_n^{(1)ab}[g;h] \equiv T^{(1)ab}[g,\hat{\Phi}_n[g];h]$$

using, as always, a Weyl ordering prescription for the operators in the last definition. Note that the third term on the right-hand side of Eq. (3.33) is due to the dependence on h_{cd} of the field operator $\hat{\Phi}_n[g+h]$ and of the dimensional regularized version of the density operator $\hat{\rho}[g+h]$.

Substituting (3.33) into (3.18), and taking into account that g_{ab} satisfies the semiclassical Einstein equation (3.14), we can write the Einstein–Langevin equation (3.18) as

$$\frac{1}{8\pi G_B}(G^{(1)ab}[g;h](x) - \Lambda_B h^{ab}(x)) - \frac{4}{3}\alpha_B D^{(1)ab}[g;h](x) - 2\beta_B B^{(1)ab}[g;h](x) - \mu^{-(n-4)}\langle \hat{T}_n^{(1)ab}[g;h](x)\rangle[g] + 2\int d^ny\ \sqrt{-g(y)}\mu^{-(n-4)}H_n^{abcd}[g](x,y)h_{cd}(y) = 2\mu^{-(n-4)}\xi_n^{ab}(x) \quad (3.35)$$

In the last equation we have used the superior index (1) to denote the terms of first order in the expansions $G_L^{ab}[g+h]$, $D_L^{ab}[g+h]$, and $B_L^{ab}[g+h]$. Thus, for instance, $G_L^{ab}[g+h] = G^{ab}[g] + G^{(1)ab}[g;h]$. The explicit expressions for the tensors $T^{(1)ab}[g,\Phi_n;h]$, $G^{(1)ab}[g;h]$, $D^{(1)ab}[g;h]$, and $B^{(1)ab}[g;h]$ are given in the Appendix. From $T^{(1)ab}[g,\Phi_n;h]$, we can write an explicit expression for the operator $\hat{T}_n^{(1)ab}$. Using the Klein–Gordon equation (3.10) and expressions (3.11) and (3.12) for the stress-energy operator, we can write this operator as

$$\hat{T}_n^{(1)ab}[g;h] = \left(\frac{1}{2}g^{ab}h_{cd} - \delta_c^a h_d^b - \delta_c^b h_d^a\right)\hat{T}_n^{cd}[g] + \mathcal{F}^{ab}[g;h]\,\hat{\Phi}_n^2[g] \quad (3.36)$$

where $\mathcal{F}^{ab}[g;h]$ is the differential operator

$$\mathcal{F}^{ab} \equiv \left(\xi - \frac{1}{4}\right)\left(h^{ab} - \frac{1}{2}g^{ab}h_c^c\right)\Box + \frac{\xi}{2}[\nabla^c\nabla^a h_c^b + \nabla^c\nabla^b h_c^a - \Box h^{ab} - \nabla^a\nabla^b h_c^c - g^{ab}\nabla^c\nabla^d h_{cd} + g^{ab}\Box h_c^c + (\nabla^a h_c^b + \nabla^b h_c^a - \nabla_c h^{ab} - 2g^{ab}\nabla^d h_{cd} + g^{ab}\nabla_c h_d^d)\nabla^c - g^{ab}h_{cd}\nabla^c\nabla^d] \quad (3.37)$$

and it is understood that indices are raised with the background inverse metric g^{ab} and that all the covariant derivatives are associated to the metric g_{ab}. Substituting expression (3.36) into Eq. (3.35) and using the semiclassical equation (3.14) to get an expression for $\mu^{-(n-4)}\langle \hat{T}_n^{ab} \rangle [g]$, we can finally write the semiclassical Einstein–Langevin equation in dimensional regularization as

$$\frac{1}{8\pi G_B}\left[G^{(1)ab} - \frac{1}{2} g^{ab} G^{cd} h_{cd} + G^{ac} h_c^b + G^{bc} h_c^a + \Lambda_B \left(h^{ab} - \frac{1}{2} g^{ab}\, h_c^c \right) \right](x)$$
$$- \frac{4}{3}\alpha_B \left(D^{(1)ab} - \frac{1}{2} g^{ab} D^{cd} h_{cd} + D^{ac} h_c^b + D^{bc} h_c^a \right)(x)$$
$$- 2\beta_B \left(B^{(1)ab} - \frac{1}{2} g^{ab} B^{cd} h_{cd} + B^{ac} h_c^b + B^{bc} h_c^a \right)(x)$$
$$- \mu^{-(n-4)} \mathcal{F}_x^{ab} \langle \hat{\Phi}_n^2(x) \rangle [g] + 2 \int d^n y \ \sqrt{-g(y)}\, \mu^{-(n-4)} H_n^{abcd}[g](x, y) h_{cd}(y)$$
$$= 2\mu^{-(n-4)} \xi_n^{ab}(x) \tag{3.38}$$

where the tensors G^{ab}, D^{ab}, and B^{ab} are computed from the semiclassical metric g_{ab}, and where we have omitted the functional dependence on g_{ab} and h_{ab} in $G^{(1)ab}$, $D^{(1)ab}$, $B^{(1)ab}$, and $\mathcal{F}^{ab}$ to simplify the notation. Notice that in Eq. (3.38) all the ultraviolet divergences in the limit $n \to 4$, which shall be removed by renormalization of the coupling constants, are in $\langle \hat{\Phi}_n^2(x) \rangle$ and the symmetric part $H_{S_n}^{abcd}(x, y)$ of the kernel $H_n^{abcd}(x, y)$, whereas, as we have pointed out above, the kernels $N_n^{abcd}(x, y)$ and $H_{A_n}^{abcd}(x, y)$ are free of ultraviolet divergences. Once we have performed such a renormalization procedure, setting $n = 4$ in this equation will yield the physical semiclassical Einstein–Langevin equation (3.25). Note that, due to the presence of the kernel $H_n^{abcd}(x, y)$ in Eq. (3.38), such an Einstein–Langevin equation will be nonlocal in the metric perturbation.

3.3. Discussion

We have seen that effective equations of motion for the metric field of the form (3.17) and (3.25) follow from the local approximation (3.2) for the effective action describing the "effective interaction" of the metric and the scalar field. A more realistic evaluation of this effective action starting from a fundamental theory of quantum gravity would certainly lead to some real and imaginary nonlocal terms in this action. In some situations, the contribution of these terms to the effective equations of motion for the metric (note that they would also give some extra terms in the semiclassical equation) might not be negligible and, in any case, one would expect that their role in the decoherence mechanism for the metric field would be important. This would represent nontrivial effects coming from the "high-momentum" modes of quantum gravity, which are not part of the gravitational field described by the classical stochastic metric $g_{ab} + h_{ab}$, but which can be a source of this gravitational field in the same way as the matter fields. The contribution of these neglected terms to the equations for the background metric g_{ab} and for the stochastic metric perturbation h_{ab} would be similar to the contribution of the scalar field through its stress-energy operator, but with this operator replaced with some "effective" stress-energy operator of such primordial "high-momentum" gravitational modes coupled to the scalar field. These equations would take the form (3.17) and (3.25) only when the effect of this "effective" stress-energy tensor on the classical spacetime geometry can be neglected. A way of partially modeling this effect would consist in replacing the stress-energy operator $\hat{T}_n^{ab}[g]$ by $\hat{T}_n^{ab}[g] + \hat{t}_n^{ab}[g]$, where $\hat{t}_n^{ab}[g]$ is the stress-energy tensor of gravitons quantized in classical spacetime background $(\mathcal{M}, g_{ab})$ [33].

We end this paper with some comments on the relation between the semiclassical Einstein–Langevin equation (3.25) and the Langevin-type equations for stochastic metric perturbations recently derived in the literature [6–8, 13–16]. In these previous derivations, one starts with the influence functional (3.4), with the state of the scalar field assumed to be an "in" vacuum or an "in" thermal state, and computes explicitly the expansion for the corresponding influence action around a specific metric background. One then applies the method of Section 2.5 to derive a Langevin equation for the perturbations to this background. As we have seen in Section 2.5, this method yields the same equations as the one used in this section. However, in most previous derivations, one starts with a "minisuperspace" model and thus the metric perturbations are assumed from the beginning to have a restrictive form. In those cases, the derived Langevin equations do not correspond exactly to our equation (3.25), but to a "reduced" version of this equation, in which only some components of the noise kernel in Eq. (3.26) (or some particular combinations of them) influence the dynamics of the metric perturbations. Only those equations which have been derived starting from a completely general form for the metric perturbations are actually particular cases, computed explicitly, of the semiclassical Einstein–Langevin equation (3.25) [13, 14, 16].

APPENDIX: EXPANSIONS AROUND A BACKGROUND METRIC

For a metric of the form $\tilde{g}_{ab} \equiv g_{ab} + h_{ab}$, where h_{ab} is a small perturbation to a background metric g_{ab}, we list the expansions of metric functionals around the background metric up to linear order in the perturbation. In the following expressions, all the tilded quantities refer to functionals constructed with the metric $\tilde{g}_{ab}$, whereas that the analogous untilded ones are constructed with the background metric g_{ab}. In particular, $\tilde{\nabla}_a$ and ∇_a are, respectively, the covariant derivatives associated to the metric $\tilde{g}_{ab}$ and to the metric g_{ab}, and $\tilde{\nabla}^a \equiv \tilde{g}^{ab}\tilde{\nabla}_b$, $\tilde{\Box} \equiv \tilde{\nabla}^a\tilde{\nabla}_a$, $\nabla^a \equiv g^{ab}\nabla_b$, $\Box \equiv \nabla^a\nabla_a$, where $\tilde{g}^{ab}$ and g^{ab} are, respectively, the inverses of $\tilde{g}_{ab}$ and g_{ab}. We shall also raise indices in the metric perturbation with the inverse background metric g^{ab}: $h^a_b \equiv g^{ac}h_{cb}$ and $h^{ab} \equiv g^{ac}\, g^{bd}\, h_{cd}$. We have

$$\tilde{g}^{ab} = g^{ab} - h^{ab} + O(h^2) \tag{A.1}$$

$$\sqrt{-\tilde{g}} = \sqrt{-g}\left(1 + \tfrac{1}{2}h^a_a + O(h^2)\right) \tag{A.2}$$

$$\tilde{\Gamma}^c_{ab} = \Gamma^c_{ab} + \tfrac{1}{2}(\nabla_a h^c_b + \nabla_b h^c_a - \nabla^c h_{ab}) + O(h^2) \tag{A.3}$$

For a scalar function f,

$$\tilde{\nabla}_a\tilde{\nabla}_b f = \nabla_a\nabla_b f - \tfrac{1}{2}\nabla^c f(\nabla_a h_{bc} + \nabla_b h_{ac} - \nabla_c h_{ab}) + O(h^2) \tag{A.4}$$

$$\tilde{\Box} f = \Box f - \nabla^a\nabla^b f\, h_{ab} - \nabla^a f(\nabla^b h_{ab} - \tfrac{1}{2}\nabla_a h^b_b) + O(h^2) \tag{A.5}$$

$$\begin{aligned}\tilde{\nabla}^a\tilde{\nabla}^b f = {} & \nabla^a\nabla^b f - \nabla^a\nabla^c f\, h^b_c - \nabla^b\nabla^c f\, h^a_c \\ & - \tfrac{1}{2}\nabla^c f(\nabla^a h^b_c + \nabla^b h^a_c - \nabla_c h^{ab}) + O(h^2)\end{aligned} \tag{A.6}$$

For a tensor t^{ab},

$$\begin{aligned}\tilde{\Box} t^{ab} = {} & \Box t^{ab} - \nabla^c\nabla^d t^{ab} h_{cd} + (g^{ae}\nabla^c t^{db} + g^{be}\nabla^c t^{ad} - \tfrac{1}{2}g^{cd}\nabla^e t^{ab}) \\ & \times (\nabla_c h_{de} + \nabla_d h_{ce} - \nabla_e h_{cd}) \\ & + \tfrac{1}{2}(g^{ac}t^{db} + g^{bc}t^{ad})(\nabla^e\nabla_d\, h_{ce} + \Box h_{cd} - \nabla^e\nabla_c h_{de}) + O(h^2)\end{aligned} \tag{A.7}$$

For the curvature tensors,

$$\tilde{R}_{ab} = R_{ab} + \tfrac{1}{2}(\nabla^c\nabla_a h_{bc} + \nabla^c\nabla_b h_{ac} - \Box h_{ab} - \nabla_a\nabla_b h^c_c) + O(h^2) \tag{A.8}$$

$$\tilde{R}^a_b = R^a_b - R^c_b h^a_c + \tfrac{1}{2}(\nabla^c\nabla_b h^a_c + \nabla^c\nabla^a h_{bc} - \Box h^a_b - \nabla_b\nabla^a h^c_c) + O(h^2) \tag{A.9}$$

$$\tilde{R} = R - R^{ab}h_{ab} + \nabla^a\nabla^b h_{ab} - \Box h^a_a + O(h^2) \tag{A.10}$$

$$\begin{aligned}\tilde{R}^{ab} = {} & R^{ab} - R^{ac}h^b_c - R^{bc}h^a_c \\ & + \tfrac{1}{2}(\nabla^c\nabla^a h^b_c + \nabla^c\nabla^b h^a_c - \Box h^{ab} - \nabla^a\nabla^b h^c_c) + O(h^2)\end{aligned} \tag{A.11}$$

$$\tilde{G}^{ab} = G^{ab} + G^{(1)ab} + O(h^2)$$

with

$$
\begin{aligned}
G^{(1)ab} &= -R^{ac}h_c^b - R^{bc}h_c^a + \tfrac{1}{2}[Rh^{ab} + g^{ab}R^{cd}h_{cd} + \nabla^c\nabla^a h_c^b \\
&\quad + \nabla^c\nabla^b h_c^a - \Box h^{ab} - \nabla^a\nabla^b h_c^c + g^{ab}(\Box h_c^c - \nabla^c\nabla^d h_{cd})] \\
&= -G^{ac}h_c^b - G^{bc}h_c^a + \tfrac{1}{2}[-Rh^{ab} + g^{ab}R^{cd}h_{cd} + \nabla^c\nabla^a h_c^b \\
&\quad + \nabla^c\nabla^b h_c^a - \Box h^{ab} - \nabla^a\nabla^b h_c^c + g^{ab}(\Box h_c^c - \nabla^c\nabla^d h_{cd})] \qquad \text{(A.12)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}^a_{bcd} &= R^a_{bcd} + \tfrac{1}{2}(\nabla_c\nabla_b h_d^a + \nabla_c\nabla_d h_b^a + \nabla_d\nabla^a h_{bc} \\
&\quad - \nabla_c\nabla^a h_{bd} - \nabla_d\nabla_b h_c^a - \nabla_d\nabla_c h_b^a) + O(h^2) \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{abcd} &= R_{abcd} + \tfrac{1}{2}(R^e_{bcd}\, h_{ae} - R^e_{acd}\, h_{be}) \\
&\quad + \tfrac{1}{2}(\nabla_c\nabla_b h_{ad} + \nabla_d\nabla_a h_{bc} - \nabla_c\nabla_a h_{bd} - \nabla_d\nabla_b h_{ac}) \\
&\quad + O(h^2) \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}^{abcd} &= R^{abcd} - \tfrac{1}{2}(2R^{abce}h_e^d + 2R^{abed}h_e^c + R^{aecd}h_e^b + R^{ebcd}h_e^a) \\
&\quad + \tfrac{1}{2}(\nabla^c\nabla^b h^{ad} + \nabla^d\nabla^a h^{bc} - \nabla^c\nabla^a h^{bd} - \nabla^d\nabla^b h^{ac}) \\
&\quad + O(h^2) \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\nabla}^a\tilde{\nabla}^b\tilde{R} &= \nabla^a\nabla^b R - \nabla^a\nabla^b(R^{cd}h_{cd}) + \nabla^a\nabla^b\nabla^c\nabla^d h_{cd} \\
&\quad - \nabla^a\nabla^b\Box h_c^c - \nabla^a\nabla^c R h_c^b - \nabla^b\nabla^c R h_c^a \\
&\quad - \tfrac{1}{2}\nabla^c R(\nabla^a h_c^b + \nabla^b h_c^a - \nabla_c h^{ab}) + O(h^2) \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Box}\tilde{R} &= \Box R - \Box(R^{ab}h_{ab}) + \Box\nabla^a\nabla^b h_{ab} - \Box^2 h_a^a - \nabla^a\nabla^b R\, h_{ab} \\
&\quad - \nabla^a R(\nabla^b h_{ab} - \tfrac{1}{2}\nabla_a h_b^b) + O(h^2) \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Box}\tilde{R}^{ab} &= \Box R^{ab} - \Box(R^{ac}h_c^b + R^{bc}h_c^a) - \nabla^c\nabla^d R^{ab}\, h_{cd} \\
&\quad - \nabla^c R^{ab}(\nabla^d h_{cd} - \tfrac{1}{2}\nabla^c h_d^d) + \nabla^c R^{ad}(\nabla_c h_d^b + \nabla_d h_c^b - \nabla^b h_{cd}) \\
&\quad + \nabla^c R^{bd}(\nabla_c h_d^a + \nabla_d h_c^a - \nabla^a h_{cd}) \\
&\quad + \tfrac{1}{2}R^{ac}(\nabla^d\nabla_c h_d^b + \Box h_c^b - \nabla^d\nabla^b h_{cd}) \\
&\quad + \tfrac{1}{2}R^{bc}(\nabla^d\nabla_c h_d^a + \Box h_c^a - \nabla^d\nabla^a h_{cd}) \\
&\quad + \tfrac{1}{2}(\Box\nabla^c\nabla^a h_c^b + \Box\nabla^c\nabla^b h_c^a - \Box^2 h^{ab} - \Box\nabla^a\nabla^b h_c^c) \\
&\quad + O(h^2) \qquad \text{(A.18)}
\end{aligned}
$$

$$
\tilde{R}^2 = R^2 - 2RR^{ab}h_{ab} + 2R\nabla^a\nabla^b h_{ab} - 2R\Box h_a^a + O(h^2) \qquad \text{(A.19)}
$$

$$\tilde{R}\tilde{R}^{ab} = RR^{ab} - RR^{ac}h_c^b - RR^{bc}h_c^a - R^{ab}R^{cd}h_{cd} + \tfrac{1}{2}R(\nabla^c\nabla^a h_c^b + \nabla^c\nabla^b h_c^a - \Box h^{ab} - \nabla^a\nabla^b h_c^c) + R^{ab}(\nabla^c\nabla^d h_{cd} - \Box h_c^c) + O(h^2) \quad \text{(A.20)}$$

$$\tilde{R}^{ab}\tilde{R}_{ab} = R^{ab}R_{ab} - 2R^{ab}R_a^c h_{bc} + R^{ab}(2\nabla^c\nabla_a h_{bc} - \Box h_{ab} - \nabla_a\nabla_b h_c^c) + O(h^2) \quad \text{(A.21)}$$

$$\tilde{R}^{ac}\tilde{R}_c^b = R^{ac}R_c^b - R^{ac}R^{bd}h_{cd} - R^{cd}(R_c^a h_d^b + R_c^b h_d^a) + \tfrac{1}{2}R^{ac}(\nabla^d\nabla_c h_d^b + \nabla^d\nabla^b h_{cd} - \Box h_c^b - \nabla_c\nabla^b h_d^d) + \tfrac{1}{2}R^{bc}(\nabla^d\nabla_c h_d^a + \nabla^d\nabla^a h_{cd} - \Box h_c^a - \nabla_c\nabla^a h_d^d) + O(h^2) \quad \text{(A.22)}$$

$$\tilde{R}^{abcd}\tilde{R}_{abcd} = R^{abcd}R_{abcd} - 2R^{abcd}R_{abce}h_d^e + 4R^{abcd}\nabla_c\nabla_b h_{ad} + O(h^2) \quad \text{(A.23)}$$

$$\tilde{R}^{acbd}\tilde{R}_{cd} = R^{acbd}R_{cd} + \tfrac{1}{2}R_{cd}(R^{acde}h_e^b + R^{bcde}h_e^a - 2R^{acbe}h_e^d - 2R^{bcae}h_e^d) + \tfrac{1}{2}R^{acbd}(\nabla^e\nabla_c h_{de} + \nabla^e\nabla_d h_{ce} - \Box h_{cd} - \nabla_c\nabla_d h_e^e) - \tfrac{1}{4}R_{cd}(2\nabla^c\nabla^d h^{ab} + \nabla^a\nabla^b h^{cd} + \nabla^b\nabla^a h^{cd} - 2\nabla^a\nabla^c h^{bd} - 2\nabla^b\nabla^c h^{ad}) + O(h^2) \quad \text{(A.24)}$$

$$\tilde{R}^{acde}\tilde{R}_{cde}^b = R^{acde}R_{cde}^b - \tfrac{1}{2}(R^{acde}R_{cde}^f h_f^b + R^{bcde}R_{cde}^f h_f^a) - 2R^{acde}R_{cdf}^b h_e^f + \tfrac{1}{2}R^{acde}(\nabla_d\nabla_c h_e^b + \nabla_e\nabla^b h_{cd} - \nabla_e\nabla_c h_d^b - \nabla_d\nabla^b h_{ce}) + \tfrac{1}{2}R^{bcde}(\nabla_d\nabla_c h_e^a + \nabla_e\nabla^a h_{cd} - \nabla_e\nabla_c h_d^a - \nabla_d\nabla^a h_{ce}) + O(h^2) \quad \text{(A.25)}$$

$$\tilde{B}^{ab} = B^{ab} + B^{(1)ab} + O(h^2)$$

with

$$B^{(1)ab} = -\tfrac{1}{2}R^2h^{ab} - g^{ab}RR^{cd}h_{cd} + 2R(R^{ac}h_c^b + R^{bc}h_c^a) + R\nabla^a\nabla^b h_c^c + 2R^{ab}(R^{cd}h_{cd} + \Box h_c^c - \nabla^c\nabla^d h_{cd}) + g^{ab}\nabla^c\nabla^d Rh_{cd} + 2\,\Box Rh^{ab}$$

$$
\begin{aligned}
&- 2\nabla^c\nabla^a R h^b_c - 2\nabla^c\nabla^b R h^a_c + g^{ab}\nabla^c\nabla^d(R h_{cd}) \\
&- \nabla^c[R(g^{ab}\nabla_c h^d_d + \nabla^a h^b_c + \nabla^b h^a_c - \nabla_c h^{ab})] \\
&+ 2g^{ab}\Box(R^{cd}h_{cd} + \Box h^c_c - \nabla^c\nabla^d h_{cd}) \\
&- 2\nabla^a\nabla^b(R^{cd}h_{cd} + \Box h^c_c - \nabla^c\nabla^d h_{cd}) \qquad \text{(A.26)}
\end{aligned}
$$

$$
\tilde{D}^{ab} = D^{ab} + D^{(1)ab} + O(h^2)
$$

with

$$
\begin{aligned}
D^{(1)ab} =& \tfrac{1}{2}(R^{cd}R_{cd} - R^{cdef}R_{cdef})h^{ab} + 2R_{cdef}(R^{acde}h^{bf} + R^{bcde}h^{af}) \\
&- R_{cd}(4R^{ac}h^{bd} + 4R^{bc}h^{ad} + R^{acde}h^b_e + R^{bcde}h^a_e - 2R^{acbe}h^d_e - 2R^{bcae}h^d_e) \\
&+ g^{ab}(R^{cf}R_{cg} - R^{cdef}R_{cdeg})h^g_f - 4R^{ac}R^{bd}h_{cd} + 4R^{acde}R^b_{cdf}h^f_e \\
&+ \tfrac{1}{2}R^{cd}(2\nabla_c\nabla_d h^{ab} + \nabla^a\nabla^b h_{cd} + \nabla^b\nabla^a h_{cd} - 2\nabla^a\nabla_c h^b_d - 2\nabla^b\nabla_c h^a_d) \\
&+ \tfrac{1}{2}R^{ac}(\nabla^d\nabla_c h^b_d - 7\Box h^b_c + 7\nabla^d\nabla^b h_{cd} - 4\nabla_c\nabla^b h^d_d) \\
&+ \tfrac{1}{2}R^{bc}(\nabla^d\nabla_c h^a_d - 7\Box h^a_c + 7\nabla^d\nabla^a h_{cd} - 4\nabla_c\nabla^a h^d_d) \\
&- R^{acde}(\nabla_d\nabla_c h^b_e + \nabla_e\nabla^b h_{cd} - \nabla_d\nabla^b h_{ce} - \nabla_e\nabla_c h^b_d) \\
&- R^{bcde}(\nabla_d\nabla_c h^a_e + \nabla_e\nabla^a h_{cd} - \nabla_d\nabla^a h_{ce} - \nabla_e\nabla_c h^a_d) \\
&- \tfrac{1}{2}g^{ab}R^{cd}(2\nabla^e\nabla_c h_{de} - \Box h_{cd} - \nabla_c\nabla_d h^e_e) \\
&+ 2g^{ab}R^{cdef}\nabla_e\nabla_d h_{cf} - R^{acbd}(\nabla^e\nabla_c h_{de} + \nabla^e\nabla_d h_{ce} - \Box h_{cd} - \nabla_c\nabla_d h^e_e) \\
&+ \tfrac{1}{2}\nabla^c R(\nabla_c h^{ab} - \nabla^a h^b_c - \nabla^b h^a_c) - 3\nabla^c R^{ad}(\nabla_c h^b_d + \nabla_d h^b_c - \nabla^b h_{cd}) \\
&- 3\nabla^c R^{bd}(\nabla_c h^a_d + \nabla_d h^a_c - \nabla^a h_{cd}) \\
&- \tfrac{1}{4}(g^{ab}\nabla^c R - 6\nabla^c R^{ab})(2\nabla^d h_{cd} - \nabla_c h^d_d) \\
&- \tfrac{1}{2}\Box R\, h^{ab} - \nabla^a\nabla^c R h^b_c - \nabla^b\nabla^c R h^a_c \\
&- \tfrac{1}{2}g^{ab}\nabla^c\nabla^d R\, h_{cd} + 3\nabla^c\nabla^d R^{ab}h_{cd} \\
&+ \tfrac{3}{2}\Box(2R^{ac}h^b_c + 2R^{bc}h^a_c + \Box h^{ab} + \nabla^a\nabla^b h^c_c - \nabla^c\nabla^a h^b_c - \nabla^c\nabla^b h^a_c)
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{1}{2}g^{ab}\Box(R^{cd}h_{cd} + \Box h_c^c - \nabla^c\nabla^d h_{cd})\\
&-\nabla^a\nabla^b(R^{cd}h_{cd} + \Box h_c^c - \nabla^c\nabla^d h_{cd})
\end{aligned}
\tag{A.27}
$$

For the stress-energy tensor functional,

$$
\begin{aligned}
T^{ab}[g, \Phi_n] \equiv{} & \nabla^a\Phi_n\nabla^b\Phi_n - \tfrac{1}{2}g^{ab}\nabla^c\Phi_n\nabla_c\Phi_n\\
& - \tfrac{1}{2}g^{ab}m^2\Phi_n^2 + \xi(g^{ab}\Box - \nabla^a\nabla^b + G^{ab})\Phi_n^2
\end{aligned}
$$

$$
T^{ab}[\tilde{g}, \Phi_n] = T^{ab}[g, \Phi_n] + T^{(1)ab}[g, \Phi_n; h] + O(h^2)
$$

with

$$
\begin{aligned}
T^{(1)ab}[g, \Phi_n; h] = {}& -T^{ac}[g, \Phi_n]h_c^b - T^{bc}[g, \Phi_n]h_c^a\\
& - \frac{1}{2}(\nabla^c\Phi_n\nabla_c\Phi_n + m^2\Phi_n^2)h^{ab} + \frac{1}{2}g^{ab}\nabla^c\Phi_n\nabla^d\Phi_n h_{cd}\\
& + \frac{\xi}{2}[-Rh^{ab} + g^{ab}R^{cd}h_{cd} + \nabla^c\nabla^a h_c^b + \nabla^c\nabla^b h_c^a - \nabla^a\nabla^b h_c^c\\
& - \Box h^{ab} + g^{ab}(\Box h_c^c - \nabla^c\nabla^d h_{cd}) + (\nabla^a h_c^b + \nabla^b h_c^a\\
& - \nabla_c h^{ab} - 2g^{ab}\nabla^d h_{cd} + g^{ab}\nabla_c h_d^d)\nabla^c\\
& + 2h^{ab}\Box - 2g^{ab}h_{cd}\nabla^c\nabla^d]\Phi_n^2
\end{aligned}
\tag{A.28}
$$

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REFERENCES

[1] L. H. Ford, *Ann. Phys.* **144**, 238 (1982).
[2] C.-I. Kuo and L. H. Ford, *Phys. Rev. D* **47**, 4510 (1993); N. G. Phillips and B.-L. Hu, *Phys. Rev. D* **55**, 6123 (1997).
[3] B.-L. Hu, *Physica A* **158**, 399 (1989).
[4] R. P. Feynman and F. L. Vernon, *Ann. Phys.* **24**, 118 (1963).
[5] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[6] E. Calzetta and B.-L. Hu, *Phys. Rev. D* **49**, 6636 (1994).
[7] B.-L. Hu and A. Matacz, *Phys. Rev. D* **51**, 1577 (1995).
[8] B.-L. Hu and S. Sinha, *Phys. Rev. D* **51**, 1587 (1995).
[9] A. O. Caldeira and A. J. Legget, *Physica A* **121**, 587 (1983).
[10] B.-L. Hu, J. P. Paz, and Y. Zhang, *Phys. Rev. D* **45**, 2843 (1992).
[11] B.-L. Hu and A. Matacz, *Phys. Rev. D* **49**, 6612 (1994).
[12] C. Greiner and B. Müller, *Phys. Rev. D* **55**, 1026 (1997).
[13] A. Campos and E. Verdaguer, *Phys. Rev. D* **53**, 1927 (1996).
[14] F. C. Lombardo and F. D. Mazzitelli, *Phys. Rev. D* **55**, 3889 (1997).
[15] A. Campos and E. Verdaguer, *Int. J. Theor. Phys.* **36**, 2525 (1997); E. Calzetta, A. Campos, and E. Verdaguer, *Phys. Rev. D* **56**, 2163 (1997).
[16] A. Campos and B.-L. Hu, *Phys. Rev. D* **58**, 125021 (1998).
[17] B.-L. Hu, J. P. Paz, and Y. Zhang, *Phys. Rev. D* **47**, 1576 (1993).
[18] A. Matacz, *Phys. Rev. D* **55**, 1860 (1997).
[19] M. Morikawa, *Phys. Rev. D* **33**, 3607 (1986); D.-S. Lee and D. Boyanovsky, *Nucl. Phys. B* **406**, 631 (1993).
[20] R. Zh. Shaisultanov, hep-th/9509154; hep-th/9512144.
[21] M. Gleiser and R. O. Ramos, *Phys. Rev. D* **50**, 2441 (1994); D. Boyanovsky, H. J. de Vega, R. Holman, D. S. Lee, and A. Singh, *Phys. Rev. D* **51**, 4419 (1995); E. Calzetta and B.-L. Hu, *Phys. Rev. D* **55**, 3536 (1997); M. Yamaguchi and J. Yokoyama, *Phys. Rev. D* **56**, 4544 (1997); S. A. Ramsey, B.-L. Hu, and A. M. Stylianopoulos, *Phys. Rev. D* **57**, 6003 (1998).
[22] J. Schwinger, *J. Math. Phys.* **2**, 407 (1961); *Phys. Rev.* **128**, 2425 (1962); P. M. Bakshi and K. T. Mahanthappa, *J. Math. Phys.* **4**, 1 (1963); **4**, 12 (1963); L. V. Keldysh, *Zh. Eksp. Teor. Fiz.* **47**, 1515 (1964) [*Sov. Phys. JETP* **20**, 1018 (1965)]; J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Massachusetts, 1970).
[23] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, *Phys. Rep.* **118**, 1 (1985); N. P. Landsman and Ch. G. van Weert, *Phys. Rep.* **145**, 141 (1987).
[24] R. B. Griffiths, *J. Stat. Phys.* **36**, 219 (1984).
[25] R. Omnès, *Rev. Mod. Phys.* **64**, 339 (1992), and references therein; R. Omnès, *The Interpretation of Quantum Mechanics* Princeton University Press, Princeton, New Jersey, 1994).
[26] M. Gell-Mann and J. B. Hartle, *Phys. Rev. D* **47**, 3345 (1993).
[27] J. B. Hartle, In *Gravitation and Quantizations*, B. Julia and J. Zinn-Justin, eds. (North-Holland, Amsterdam, 1995), and references therein; gr-qc/9304006.
[28] J. J. Halliwell, In *Stochastic Evolution of Quantum States in Open Systems and Measurement Processes*, L. Diósi, ed. (World Scientific, Singapore, 1994), gr-qc/9308005.
[29] J. J. Halliwell, *Ann N. Y. Acad. Sci.* **755**, 726 (1995), and references therein.
[30] J. P. Paz and W. H. Zurek, *Phys. Rev. D* **48**, 2728 (1993).
[31] H. F. Dowker and J. J. Halliwell, *Phys. Rev. D* **46**, 1580 (1992).
[32] J. J. Halliwell, *Phys. Rev. D* **48**, 4785 (1993); **57**, 2337 (1998).
[33] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[34] E. S. Abers and B. W. Lee, *Phys. Rep.* **9C**, 1 (1973).
[35] S. Weinberg, *The Quantum Theory of Fields*, Vols. I and II (Cambridge University Press, Cambridge, 1995, 1996).
[36] V. Hakim and V. Ambegaokar, *Phys. Rev. A* **32**, 423 (1985); C. Morais Smith and A. O. Caldeira, *Phys. Rev. A* **36**, 3509 (1987).
[37] H. Grabert, P. Schramm, and G.-L. Ingold, *Phys. Rep.* **168**, 115 (1988).
[38] Z.-B. Su, L.-Y. Chen, X.-T. Yu, and K.-C. Chou, *Phys. Rev. B* **37**, 9810 (1988).

[39] W. H. Zurek, In *Conceptual Problems of Quantum Gravity*, A. Ashtekar and J. Stachel, eds. (Birkhäuser, Boston, 1991); *Phys. Today* **44**(10), 36 (1991); *Vistas Astron.* **37**, 185 (1993).
[40] T. Brun, *Phys. Rev. D* **47**, 3383 (1993); J. P. Paz, S. Habib, and W. H. Zurek, *Phys. Rev. D* **47**, 488 (1993).
[41] F. Lombardo and F. D. Mazzitelli, *Phys. Rev. D* **53**, 2001 (1996); T. Tanaka and M. Sakagami, *Prog. Theor. Phys.* **100**, 547 (1998).
[42] J. T. Whelan, *Phys. Rev. D* **57**, 768 (1998); qr-qc/9702003.
[43] E. Joos and H. D. Zeh, *Z. Phys. B* **59**, 223 (1985).
[44] E. Wigner, *Phys. Rev.* **40**, 749 (1932); M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984); J. J. Halliwell, *Phys. Rev. D* **36**, 3626 (1987).
[45] J. F. Donoghue, *Phys. Rev. Lett.* **72**, 2996 (1994); *Phys. Rev. D* **50**, 3874 (1994); *Helv. Phys. Acta* **69**, 269 (1996); in *Advanced School on Effective Theories*, F. Cornet and M. J. Herrero, eds. (World Scientific, Singapore, 1996), gr-qc/9512024; gr-qc/9712070.
[46] T. S. Bunch, *J. Phys. A* **12**, 517 (1979).
[47] R. Martín and E. Verdaguer, gr-qc/9811070; *Phys. Rev. D* **60**, 084008 (1999).